NOTE CAREFULLY

The following document was developed by

Centre for Learning Innovation, DET.

This material does not contain any 3rd party copyright items. Consequently, you may use this material in any way you like providing you observe moral rights obligations regarding attributions to source and author. For example:

This material was adapted from ‘(Title of CLI material)’ produced by Centre for Learning Innovation, DET.
Algebraic Modelling
Acknowledgments

This publication is copyright Learning Materials Production, Open Training and Education Network – Distance Education, NSW Department of Education and Training, however it may contain material from other sources which is not owned by Learning Materials Production. Learning Materials Production would like to acknowledge the following people and organisations whose material has been used.

- Extract from General Mathematics Stage 6 Syllabus © Board of Studies NSW, originally issued 1999. The most up-to-date version can be found on the Board’s website at http://www.boardofstudies.nsw.edu.au/syllabus_hsc/syllabus2000_listm.html#m

All reasonable efforts have been made to obtain copyright permissions. All claims will be settled in good faith.

Writer: Jim Stamell
Editor: Teresa Ashton, Kristine Turbet
Revised: Margaret Willard
Illustrator: Tim Hutchinson, Barbara Gurney, Thomas Brown
Cover illustration: Thomas Brown
Layout: Gayle Reddy

Copyright in this material is reserved to the Crown in the right of the State of New South Wales. Reproduction or transmittal in whole, or in part, other than in accordance with provisions of the Copyright Act, is prohibited without the written authority of Learning Materials Production.

Contents

Module overview ........................................................................................................ iii

Outcomes ................................................................................................................ iv

Indicative time ........................................................................................................ iv

Course overview .................................................................................................. v

Icons ....................................................................................................................... vi

Formulae sheet .................................................................................................... vii–viii

AM3: Algebraic skills and techniques

Part 1: Algebraic skills ...................................................................................... 1 - 48

AM4: Modelling linear and non-linear relationships

Part 1: Lines and parabolas ................................................................................ 1 - 52

Part 2: Non-linear graphs .................................................................................... 1 - 104

Part 3: Variation .................................................................................................. 1 – 21

Introduction
This module addresses the Algebraic Modelling Area of Study, and has two sections, AM3 and AM4, which are subdivided into parts.

AM3: Algebraic skills and techniques. This unit develops algebraic skills and techniques that are used in work-related and everyday contexts. As far as possible, real contexts should be used to demonstrate the use of algebra in practical life.

AM4: Modelling linear and non-linear relationships. This unit focuses on the examination of practical problems that can be modelled algebraically.

The relative importance of Algebraic Modelling in relation to the whole HSC course is demonstrated in the Course overview. Note that the order of listing is by topic, and is not necessarily a prescribed sequence in which students will study the whole course.
Outcomes

Within the various parts of this module each of the following outcomes will be addressed to some extent.

A student

H1 appreciates the importance of mathematics in his/her own life and its usefulness in contributing to society

H2 integrates mathematical knowledge and skills from different content areas in exploring new situations

H3 develops and tests a general mathematical relationship from observed patterns

H5 makes predictions about the behaviour of situations based on simple models

H7 interprets the results of measurements and calculations and makes judgements about reasonableness

H11 uses mathematical argument and reasoning to evaluate conclusions drawn from other sources, communicating his/her position clearly to others.

Extract from General Mathematics Stage 6 Syllabus © Board of Studies NSW, originally issued 1999. The most up-to-date version can be found on the Board's website at http://www.boardofstudies.nsw.edu.au/syllabus_hsc/syllabus2000_listm.html#m

Indicative time

Students should allocate up to 24 of the total 120 course hours to this area of study, allowing perhaps up to 5 hours for each part, and leaving time for revision and set assessment tasks or examinations.
Course overview

**Financial Mathematics**
- FM4 Credit and borrowing
- FM5 Annuities and loan repayments
- FM6 Depreciation

**Data Analysis**
- DA5 Interpreting sets of data
- DA6 The normal distribution
- DA7 Correlation

**Measurement**
- M5 Further applications of area and volume
- M6 Applications of trigonometry
- M7 Spherical geometry

**Probability**
- PB3 Multi–stage events
- PB4 Applications of probability

**Algebraic Modelling**
- AM3 Algebraic skills and techniques
- AM4 Modelling linear and non–linear relationships
The following icons are used within this module. The meaning of each icon is written beside it.

- There is an activity for you to do. It may be you need to collect data or you may make something.
- You need to use a calculator for this activity.
- There are example with solutions for you to work through.
- There is an exercise for you to complete.

**FORMULAE SHEET**

**Area of an annulus**
\[ A = \pi (R^2 - r^2) \]

\( R = \) radius of outer circle  
\( r = \) radius of inner circle

**Area of an ellipse**
\[ A = \pi ab \]

\( a = \) length of semi-major axis  
\( b = \) length of semi-minor axis

**Area of a sector**
\[ A = \frac{\theta}{360} \pi r^2 \]

\( \theta = \) number of degrees in central angle

**Arc length of a circle**
\[ l = \frac{\theta}{360} 2\pi r \]

\( \theta = \) number of degrees in central angle

**Simpson’s rule for area approximation**
\[ A \approx \frac{h}{3} (d_f + 4d_m + d_l) \]

\( h = \) distance between successive measurements  
\( d_f = \) first measurement  
\( d_m = \) middle measurement  
\( d_l = \) last measurement

**Surface area**
- **Sphere**  
  \[ A = 4\pi r^2 \]
- **Closed cylinder**  
  \[ A = 2\pi rh + 2\pi r^2 \]

\( r = \) radius  
\( h = \) perpendicular height

**Volume**
- **Cone**  
  \[ V = \frac{1}{3} \pi r^2 h \]
- **Cylinder**  
  \[ V = \pi r^2 h \]
- **Pyramid**  
  \[ V = \frac{1}{3} Ah \]
- **Sphere**  
  \[ V = \frac{4}{3} \pi r^3 \]

\( r = \) radius  
\( h = \) perpendicular height  
\( A = \) area of base

**Sine rule**
\[ \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \]

**Area of a triangle**
\[ A = \frac{1}{2} ab \sin C \]

**Cosine rule**
\[ c^2 = a^2 + b^2 - 2ab \cos C \]

or
\[ \cos C = \frac{a^2 + b^2 - c^2}{2ab} \]
FORMULAE SHEET

Simple interest

\[ I = P r n \]

- \( P \) = initial quantity
- \( r \) = percentage interest rate per period, expressed as a decimal
- \( n \) = number of periods

Declining balance formula for depreciation

\[ S = V_0 (1 - r)^n \]

- \( S \) = salvage value of asset after \( n \) periods
- \( r \) = percentage interest rate per period, expressed as a decimal

Compound interest

\[ A = P (1 + r)^n \]

- \( A \) = final balance
- \( P \) = initial quantity
- \( n \) = number of compounding periods
- \( r \) = percentage interest rate per compounding period, expressed as a decimal

Mean of a sample

\[ \bar{x} = \frac{\sum x}{n} \]

- \( x \) = individual score
- \( \bar{x} \) = mean
- \( n \) = number of scores
- \( f \) = frequency

Future value (\( A \)) of an annuity

\[ A = M \left( \frac{(1 + r)^n - 1}{r} \right) \]

- \( M \) = contribution per period, paid at the end of the period

Formula for a \( z \)-score

\[ z = \frac{x - \bar{x}}{s} \]

- \( s \) = standard deviation

Present value (\( N \)) of an annuity

\[ N = M \left( \frac{(1 + r)^n - 1}{r(1 + r)^n} \right) \]

or

\[ N = \frac{A}{(1 + r)^n} \]

Gradient of a straight line

\[ m = \frac{\text{vertical change in position}}{\text{horizontal change in position}} \]

Gradient–intercept form of a straight line

\[ y = mx + b \]

- \( m \) = gradient
- \( b \) = \( y \)-intercept

Probability of an event

The probability of an event where outcomes are equally likely is given by:

\[ P(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} \]
AM 3 Algebraic skills and techniques

Part 1: Algebraic skills
Contents

Introduction .......................................................................................................................... 3

1.1 Algebraic expressions ................................................................................................. 4
    Substituting into algebraic expressions .................................................................... 4

1.2 Manipulating expressions ......................................................................................... 8
    Adding and subtracting like terms ........................................................................ 8
    Multiplying and dividing algebraic terms .......................................................... 10

1.3 Review of simple equations ..................................................................................... 13
    Equations involving fractions ............................................................................ 15
    Squares and square roots ................................................................................ 17

1.4 Changing the subject of equations .......................................................................... 19

1.5 Substituting into formulas ....................................................................................... 21
    Harder substituting .......................................................................................... 23

1.6 Equations from practical situations .......................................................................... 27

Terminology ....................................................................................................................... 33

Exercises ............................................................................................................................ 35

Student evaluation ........................................................................................................... 39

Appendix ............................................................................................................................ 41

Answers .............................................................................................................................. 43

Part 1: Algebraic skills
Introduction

This is a single part covering the syllabus topic AM3 Algebraic skills and techniques, in the Algebraic modelling component of the course.

Specific content outcomes

By the end of Part 1, you will have been given opportunities to:

• substitute into and evaluate algebraic expressions – linear, quadratic, cubic, as well as those involving square and cube roots
• add and subtract like terms
• multiply and divide algebraic terms and expressions
• change the subject of equations and formulas involving linear and quadratic terms
• solve equations after substituting values
• solve equations arising from practical situations by estimation and refinement
• use positive and negative powers of ten as part of expressing measurements in scientific notation

For students in Distance Education Centres only:
There is an evaluation page at the back of this part; fill it in when you have finished the work; say how easy/hard/interesting you find this work; ask relevant questions and return your comments to your teacher.
The basic ideas of algebra were covered in earlier units on Algebraic Modelling in the Preliminary course. For this HSC course we will review and extend on those basic ideas. You may benefit by reviewing the basic algebraic skills if it is some time since you did the preliminary course.

You may like to know that the word ‘algebra’ comes from ‘al–jabr’ which means ‘the reunion’. It was in 830AD that the brilliant Arab scholar Al’Khwarizmi wrote Hisab al–jabr w’almuqabala where he set out some basic ideas of algebra although no symbols were used.

### Substituting into algebraic expressions

Algebra involves pronumerals as well as numerals. A pronumeral takes the place of a numeral so \( m \) is the pronumeral in the expression \( 2m + 1 \). If \( m = 4, \ 2m + 1 = 2 \times 4 + 1 = 9 \) and if \( m = -3, \ 2m + 1 = 2 \times -3 + 1 = -5 \).

The process of replacing a pronumeral by a numeral is substitution.

Find the value of the expressions given that: \( p = 4, \ g = -3, \) and \( r = 2 \frac{1}{2} \)

1. \( qp - 3r \)
2. \( 0.5qr^3 \)
3. \( \sqrt{p^2 + q^2} \)
4. \( \frac{p - q}{2q + 4r} \)
5. \( \frac{3r}{\sqrt{2\pi}} \)
6. \( \frac{1}{p} - \frac{1}{r} \)

**Solutions:**
1. \( q \cdot p - 3r = 4 \times -3 - 3 \times 2 \frac{1}{2} \)
   \[= -12 - 7 \frac{1}{2} \]
   \[= -19 \frac{1}{2} \]

2. \( 0.5qr^3 = 0.5 \times -3 \times \left(2 \frac{1}{2}\right)^3 \)
   \[= 0.5 \times -3 \times 15.625 \]
   \[= -23.4375 \]

3. \( \sqrt{p^2 + q^2} = \sqrt{4^2 + (-3)^2} \)
   \[= \sqrt{16 + 9} \]
   \[= \sqrt{25} \]
   \[= 5 \]

4. \( \frac{p - q}{2q + 4r} = \frac{4 - (-3)}{2(-3) + 4 \times 2 \frac{1}{2}} \)
   \[= \frac{7}{4} \]
   \[= 1 \frac{3}{4} \]

5. \( \sqrt[3]{\frac{3r}{2\pi}} = \sqrt[3]{\frac{3 \times 2 \frac{1}{2}}{2 \times \pi}} \)
   \[= \sqrt[3]{7.5} \div (2 \times \pi) \]
   \[\approx 1.06 \]

Note: If you have not dealt with numbers in scientific notation (standard form) for some time, you should revise M1: Units of Measurement in the Preliminary Course before proceeding further.

If \( a = 3.51 \times 10^{-5} \) and \( b = 2.78 \times 10^4 \), find the value of \( \frac{a^2}{b} \).
Give your answer in scientific notation correct to 3 significant figures.

Solution:
\[ \frac{a^2}{b} = \frac{(3.51 \times 10^{-5})^2}{2.78 \times 10^4} \]
\[= 4.43 \times 10^{-14} \]

Calculator steps are \[ \text{[c]} \; 3.51 \; \text{EXP} \; -5 \; \text{[c]} \; \times^2 \; \frac{2.78 \; \text{EXP} \; 4 \; \text{=}} \]
Note: Some answers have been given in decimal form and others in fraction form. Unless specifically asked, answers can be given as fractions or decimals, however, it is usual to give answers in decimal form if the question involves decimals and in fraction form if the question involves only fractions.

Exercise 1.1

1 If \( a = 2, \ b = 1.2 \) and \( c = -3 \), evaluate these expressions correct to three decimal places if necessary:

   a \( 3a - 2b \)
   b \( abc \)
   c \( a^2 + b^2 - c^2 \)
   d \( \frac{a + b}{b - c} \)
   e \( \sqrt[3]{\frac{a^3}{b - c}} \)
   f \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \)
   g \( 2c^3 - a^2b \)
   h \( \frac{3}{b^2c} \)
   i \( 8 - \frac{\pi}{ab} \)

2 Evaluate 2' when \( x = -3 \).

3 Calculate the value of \((y - 1)(2y + 3)\) when \( y = \frac{1}{2} \).

4 If \( y = 5x^2 - 7x - 6 \) find \( y \) when \( x = -2 \).

5 If \( h = 30t^2 - 5t^2 \), find \( h \) when \( t = 4 \).

6 If \( T = \frac{mv^2}{r} \), find \( T \) when \( m = 3.2, \ v = 7.3 \) and \( r = 9.5 \).

   Give your answer correct to one decimal place.

7 The density \( D \) g/cm\(^3\) of a body of mass \( M \) g and volume \( V \) cm\(^3\) is given by \( D = \frac{M}{V} \). Find the density of oil in g/cm\(^3\) if 769g occupies a volume of one litre. (1L = 1000cm\(^3\))
8 The formula \( f = \frac{(M - m)}{M + m} g \) gives the acceleration, \( f \), of a simple pulley system with masses \( M \) and \( m \) at the ends and where \( g = 9.8 \). Find the value of \( f \) if \( M = 23 \) and \( m = 17 \).

9 Find the value of \( 2\pi \sqrt{\frac{l}{g}} \) when \( l = 2.7 \) and \( g = 9.8 \). Give your answer correct to four significant figures.

10 The base length of a square pyramid of volume \( V \) and perpendicular height \( h \) is given by \( l = \sqrt[3]{\frac{3V}{h}} \). Find the base length, \( l \), correct to one decimal place if \( V = 574 \) and \( h = 10.4 \).

11 Calculate the value of \( \frac{1}{2} \epsilon^\tau \) in scientific notation and in ordinary decimal form if \( \epsilon = 3 \times 10^5 \) and \( \tau = 8 \times 10^{-3} \).

12 The radius, \( r \), of a sphere with volume, \( V \), can be found from \( r = \sqrt[3]{\frac{3V}{4\pi}} \). Find the radius of a sphere with volume 10cm\(^3\).

13 The new volume, \( V' \), of a gas is \( V' = V_0(1 + \alpha T) \) where \( V_0 \) litres is the original volume, \( T \) is the change in temperature and \( \alpha \) is the constant 0.00366. Find the new volume of a gas initially occupying 22.4L if the temperature changes by 85°C.
Adding and subtracting like terms

There are three different animals in this picture: bulls, lions and crabs. If you were asked to count them, you would group the bulls, the lions and the crabs to get: 6 bulls + 3 lions + 4 crabs.

Similarly, in algebra, we group like terms before we add or subtract. Like terms are terms that contain exactly the same variable. For instance, $3x^2yz$ and $8x^2yz$ are like terms but $3x^2$ and $8x$ are not.

Simplify the following by collecting like terms.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$17a + 6a - 5a$</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>$23pq + 14pq - 10pq^2$</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>$8x - 3y + 4x - 2y$</td>
<td></td>
</tr>
</tbody>
</table>

Solutions:

1. $18a$ All terms are like terms
2. $12m - 4m + 2n + 7n = 8m + 9n$ Re-arrange terms before adding
3. $37pq - 10pq^2$ $pq$ and $qp$ are like terms but $pq$ and $pq^2$ are different variables
4. $4c - c + 3c^2 + c^2 = 3c + 4c^2$ $c$ and $c^2$ are different variables
5. $8x + 4x - 3y - 2y = 12x - 5y$
Find the perimeter of the triangle.

Solution:

Perimeter = \((3a - 2) + (3a - 2) + (4a + 5)\)
= \(3a + 3a + 4a - 2 - 2 + 5\)
= \(10a + 1\)

Exercise 1.2a

1 Simplify, where possible, by collecting like terms.

a \(3d + 4d - d\)  
b \(4x^2 + 2x^2 + x^2 - x^2\)  
c \(7sa - 4as + 5s\)  
d \(2w - 4r + 2w - w - r^2\)  
e \(10wer - 9rew\)  
f \(5z^2c + 4ze^2 - 6z^2c + ze^2\)  
g \(4mk - 4mk\)  
h \(6t - (-t) + 5\)  
i \(v^2 + 5 - 2 - v^2\)  
j \(3u^4 + 5u^3 - 2u^4 - u^3\)

2 Find the perimeter of these shapes. (Lengths are in centimetres.)

a

\[
\begin{align*}
4a + 1 & \quad 3a - 2 \\
\end{align*}
\]

b

\[
\begin{align*}
f - 1 & \quad f + 2 \\
\end{align*}
\]

c

\[
\begin{align*}
2d - 1 & \quad 3d + 2 \\
\end{align*}
\]

d

\[
\begin{align*}
p + 3 \\
\end{align*}
\]

e

\[
\begin{align*}
m + 3 & \quad m - 2 \\
\end{align*}
\]

f

\[
\begin{align*}
3d - 4 & \quad 2d \\
\end{align*}
\]
3. a) Draw a square having perimeter $4y + 12$ and label a side.
   b) Draw an equilateral triangle with perimeter $6g^2 - 9$, showing the length of each side.

4. A triangular paddock has its shortest side $h$ m long. The longest side is twice the shortest side and the third side is 20m shorter than the longest side. What is the perimeter of the paddock?

**Multiplying and dividing algebraic terms**

Review multiplying and dividing algebraic terms in the Preliminary Course *AM1: Basic algebraic skills* if necessary before proceeding further. The examples below should help to refresh your memory.

Simplify the following expressions.

1. $17a \times 6a$
2. $12mn^2 + 3m^2n$
3. $23p^3q + 14qp \times 2p^2$
4. $(4ac^2 + 3a^2c) + ac$
5. $\frac{(4y)^2}{2y^2}$
6. $a^2b \times ab^2 \times ab$
7. $4(x + 2y) - 3(x + y)$
8. $16 - (x - 1)$

**Solutions:**

1. $17 \times 6 \times a \times a = 102a^2$  
   **Multiply numerals and pronumerals**

2. $12mn^2 + 3m^2n$
   $= \frac{12 \times m \times n \times n}{3 \times m \times m \times n}$
   $= \frac{4n}{m}$
   **Divide top and bottom by 3mn, the highest common factor HCF**

3. $23p^3q + 14qp \times 2p^2$
   $= 23p^3q + 28p^3q$
   $= 51p^3q$  
   **Do the multiplication before the addition then add like terms**
4 \[ (4ac^2 + 3ac^2) + ac \]
\[ = 7ac^2 + ac \]
\[ = \frac{7ac^2}{ac} \]
\[ = 7c \]

Do the grouping symbols first then divide top and bottom by the HCF \( ac \).

5 \[ \frac{(4y)^2}{2y^2} = \frac{4y \times 4y}{2 \times y^2} \]
\[ = \frac{16 \times y^2}{2 \times y^2} \]
\[ = 8 \]

Divide top and bottom by the HCF \( 2y^2 \).

6 \[ a^2 \times a \times a \times b \times b \times b \times b \]
\[ = a^4 b^4 \]

Multiply ‘a’s then ‘b’s.

7 \[ 4(x + 2y) - 3(x + y) \]
\[ = 4x + 8y - 3x - 3y \]
\[ = x + 5y \]

Both \( x \) and \( y \) in the second bracket are multiplied \(-3\).

8 \[ 16 - (x - 1) \]
\[ = 16 - x + 1 \]
\[ = 17 - x \]

Both \( x \) and \(-1\) in the bracket are multiplied by \(-1\) and \(-1 \times -1 = +1\).

**Exercise 1.2b**

1 Simplify these products.

a \[ 3d \times 4d \times d \]

b \[ 4x^2 \times 2x^2 + x^2 \times x^2 \]

c \[ 7sa \times 4as \times 5s \]

d \[ 2w \times 4r \times 2w \times w \times r^2 \]

e \[ 10wer \times 9rew \]

f \[ 5z^2 c \times 4zc^2 \times 6z^2c \times ze^2 \]

2 Simplify these quotients.

a \[ 32d \div 4 \]

b \[ 32d \div 4d \]

c \[ 12sa \div 4as \]

d \[ 20wr \div 2w \]

e \[ 18wer \div 9re \]

f \[ 5z^2c \div 4zc^2 \]
3 Simplify these expressions.

\[a\] \(3d \times 4d + 2d\)
\[b\] \(4x^2 \times 2x^2 + x^2 + x^2\)
\[c\] \(7sa \times 4as + 5s\)
\[d\] \(2u(4r + 2w) + w \times r\)
\[e\] \(10wer \times 6rew + 5w^2\)
\[f\] \(8z^2c + 4zc^2 + 6z^2c + zc^2\)
\[g\] \(-28sqr + 7sq\)
\[h\] \(12a^3 b^2 + 12a^2 b\)

4 Write the simplest expression.

\[a\] \((3d + 4d) \times 5\)
\[b\] \((4x^2 + 2x^2) + x^2 + x^2\)
\[c\] \(15sa - 4a \times 5s\)
\[d\] \(2w \times 4r - 2w \times r\)
\[e\] \(\frac{5u - u}{3u + u}\)
\[f\] \(\frac{5x \times 4y \times 2z}{10z \times y \times 12z}\)
\[g\] \(\frac{(-2g) \times (-3)}{(-4gh) \times 5}\)
\[h\] \(\frac{1}{4} y^2 z \times 8yz\)

5 Expand, collect like terms and simplify.

\[a\] \(3(3d + 4d) + 5(2d - 3e)\)
\[b\] \(m(m - 2) + 3m(2m - 3)\)
\[c\] \(15 - 2(6 - b)\)
\[d\] \(x(x + y) - x(x - y)\)
\[e\] \(5[x + 3(2x + 4)]\)
\[f\] \(2[3(y^2 + 1) - 2(2y^2 - 6)]\)

(Hint: for part e and f, expand inner brackets first)

6 What is the difference between \((-3d)^2\) and \(-3d^2\)?

7 Find the area of these shapes using appropriate formulae.
(Lengths are in centimetres.)

\[a\]
\[b\]
\[c\]
\[d\]
You will review simple equations before solving more difficult ones.

**Rule**

What you do to one side of an equation you must do to the other side so that both sides remain equal.

Solve:

1. \(2y + 4 = 9\)
2. \(2q + 5 = 3 - q\)
3. \(3(f - 4) = 4f + 5\)
4. \(5 - 4x = 3x + 5\)

**Solutions:**

1. \(2y + 4 = 9\)
   - subtract 4 from both sides
   - \(2y = 5\)
   - divide both sides by 2.
   - \(y = 2.5\)
2 \quad 2q + 5 = 3 - q \quad \text{add } q \text{ to both sides}
3q + 5 = 3 \quad \text{take } 5 \text{ from both sides}
3q = -2 \quad \text{divide by } 3
q = -\frac{2}{3}

3 \quad 3(f - 4) = 4f + 5 \quad \text{remove parentheses}
3f - 12 = 4f + 5 \quad \text{subtract } 4f \text{ from both sides}
3f - 4f - 12 = 5 \quad \text{collect like terms}
-f - 12 = 5 \quad \text{add } 12 \text{ to both sides}
-f = 5 + 12
-f = 17
f = -17

4 \quad 5 - 4x = 3x + 5 \quad \text{add } 4x \text{ to both sides}
5 = 7x + 5 \quad \text{subtract } 5 \text{ from both sides}
0 = 7x \quad \text{divide both sides by } 7
x = 0

Exercise 1.3a

Solve each of the following equations, giving answers in decimals where necessary.

1 \quad 7a = 56
2 \quad 3b + 5 = 14
3 \quad 5c + 9 = 2c - 15
4 \quad 7d + 6 = 3d - 2
5 \quad 4e - 1 = e + 8
6 \quad 2(f + 5) = 4f + 3
7 \quad 7 - 4g = 3g + 21
8 \quad 2h + 6 = 3(h - 5) + 4
9 \quad 3(i + 2) = 5i - 12
10 \quad 2(3j + 1) - 5 = 4j - (2 + 3j)
Equations involving fractions

Solve:

1. \( \frac{4z - 1}{3} = -7 \)
2. \( \frac{m}{2} - 1 = 5 \)
3. \( \frac{a}{4} - \frac{a}{3} = 3 \)

Solutions:

1. \( \frac{4z - 1}{3} = -7 \)
   - multiply both sides by 3
   - add 1 to both sides
   - divide both sides by 4
   - \( z = -5 \)

2. \( \frac{m}{2} - 1 = 5 \)
   - add 1 to both sides
   - multiply both sides by 2
   - \( m = 12 \)

3. \( \frac{a}{4} - \frac{a}{3} = 3 \)
   - multiply each term by 12 (the LCD)
   - simplify terms
   - \( 3a - 4a = 36 \)
   - \( -a = 36 \)
   - \( a = -36 \)
   - divide both sides by \(-1\)

Note: The LCD is the smallest number that each of the denominators will divide into evenly.
Exercise 1.3b

Solve each of the following equations:

1. \(\frac{x + 3}{4} = 2\)
2. \(\frac{p - 1}{2} = 6\)
3. \(\frac{d}{4} + 1 = 3\)
4. \(\frac{y}{3} + \frac{1}{3} = 1\)
5. \(\frac{3y}{5} + 2 = 8\)
6. \(m + \frac{2m}{3} = 10\)
7. \(\frac{5k + 1}{4} = 9\)
8. \(\frac{z}{5} - 4 = 6\)
9. \(\frac{3(c - 1)}{4} = 3\)
10. \(\frac{4(2y + 3)}{3} = 10\)

Harder equations involving fractions

Solve:

1. \(\frac{m}{3} - \frac{m}{6} = 4\)
2. \(\frac{y + 1}{3} + \frac{y + 2}{2} = 3\)

Solutions:

1. \(6 \times \frac{m}{3} - 6 \times \frac{m}{6} = 4 \times 6\) multiply each term by 6 (the LCD)
   \(2m - m = 24\) simplify terms
   \(m = 24\)
2. \(6 \times \frac{(y + 1)}{3} + 6 \times \frac{(y + 2)}{2} = 6 \times 3\) multiply each term by 6 (the LCD)
   \(2(y + 1) + 3(y + 2) = 18\) simplify terms
   \(2y + 2 + 3y + 6 = 18\) remove parentheses
   \(5y + 8 = 18\) simplify terms
   \(5y = 10\)
   \(y = 2\)
Exercise 1.3c

Solve each of the following equations, giving answers in fraction form where necessary.

1. \( \frac{2a}{3} + \frac{a}{2} = 4 \)
2. \( \frac{w}{4} + 3 = \frac{w}{2} \)
3. \( \frac{x-4}{2} = \frac{x+5}{3} \)
4. \( \frac{r-3}{4} = \frac{r-2}{5} \)
5. \( \frac{u}{4} + \frac{u-5}{3} = 10 \)
6. \( \frac{x+4}{5} + \frac{x-1}{2} = 1 \)

Squares and square roots

- To remove a square root, square both sides
- To remove a square, take the square root of both sides
- Similar rules apply for cubes and cube roots

Solve

1. \( \sqrt{\frac{z}{8}} = 4 \)
2. \( \frac{3y^2}{2} = 54 \)

Solutions:

1. \( \sqrt{\frac{z}{8}} = 4 \)
   
   square both sides

   \( \frac{z}{8} = 16 \)
   
   \( z = 16 \times 8 \)
   
   \( z = 128 \)

2. \( \frac{3y^2}{2} = 54 \)
   
   \( 3y^2 = 54 \times 2 \)
   
   \( y^2 = \frac{54 \times 2}{3} \)
   
   \( y^2 = 36 \)
   
   take square root of both sides
   
   \( y = \pm 6 \)
Note: There are two solutions to the second equation, +6 and –6, since 
(+6)^2 = 36 and (–6)^2 = 36.

Exercise 1.3d

Solve, giving any answers correct to two decimal places if necessary.

1 \[4x^2 = 20 - x^2\]  
2 \[2(x^2 + 5) = 28\]  
3 \[\sqrt[3]{x} = 7\]  
4 \[\sqrt{10x - 4} = 6\]  
5 \[4m^2 - 1 = 8 + m^2\]  
6 \[\sqrt{3y^2 - 10} = y\]  
7 \[\sqrt[3]{3w - 5} = 4\] [Hint: cube both sides]

Mixed revision exercises

You can do this exercise now or leave for later revision.

Exercise 1.3e

Solve:

1 \[4x = 13 - x\]  
2 \[2(x + 5) = 4\]  
3 \[\frac{x}{3} = \frac{1}{12}\]  
4 \[10x - 4x = 36 - 12\]  
5 \[\sqrt{2x - 6} = 3\]  
6 \[\sqrt[3]{x} = 6\]  
7 \[56 = x^2 - 24\]  
8 \[2(5x - 13) = 3(x + 18)\]  
9 \[5r - 1 = 2r + 5\]  
10 \[x = 2(2 - x)\]  
11 \[2x - 5 = \frac{x + 2}{2}\]  
12 \[5(x - 1) - 2(x + 4) = 4\]  
13 \[4 - 3x = 79\]  
14 \[2(3 - x) = 79\]  
15 \[\frac{(x + 4)}{3} - \frac{(x - 3)}{2} = 1\]  
16 \[7x - 5(x - 2) = 8\]  
17 \[\frac{x - 1}{3} - \frac{x - 1}{4} = 2\]  
18 \[\frac{y}{8} - \frac{1}{2} = 4\]
1.4 Changing the subject of equations

\( F \) is the subject of the formula \( F = ma. \)

If the formula is rearranged to get \( m = \frac{F}{a}, \) the subject is now \( m. \)

If the formula is rearranged to get \( a = \frac{F}{m}, \) the subject is now \( a. \)

What is the subject of each formula?

1 \( v = u + at \)
2 \( I = Ab^2 + b \)
3 \( A = \pi r^2 \)
4 \( v^2 = u^2 + 2as \)

**Solutions:**

1 \( v \) 2 \( I \) 3 \( A \) 4 \( v^2 \)

Make \( s \) the subject of the formula \( v^2 = u^2 + 2as \)

**Solution:**

\[
\begin{align*}
  v^2 &= u^2 + 2as \\
  v^2 - u^2 &= 2as \\
  2as &= v^2 - u^2 \\
  s &= \frac{v^2 - u^2}{2a}
\end{align*}
\]

Subtract \( u^2 \) from both sides to get the term in \( s \) by itself
Put term in \( s \) on the left hand side
Divide both sides by \( 2a \) (the coefficient of \( s \))
Change the subject of the formula \( A = \frac{1}{2} h(x + y) \) to \( y \).

**Solution:**

\[
A = \frac{1}{2} h(x + y) \\
2A = h(x + y) \quad \text{Multiply both sides by } 2 \text{ to remove fractions} \\
\frac{2A}{h} = x + y \quad \text{Divide both sides by } h \\
y = \frac{2A}{h} - x \quad \text{Subtract } x \text{ from both sides}
\]

**Alternative solution**

\[
2A = h(x + y) \\
2A = hx + hy \\
2A - hx = hy \quad \text{Subtract } hx \text{ from both sides} \\
y = \frac{2A - hx}{h} \quad \text{Divide both sides by } h
\]

**Note:** Both results are the same since \( \frac{2A - hx}{h} = \frac{2A}{h} - \frac{hx}{h} = \frac{2A}{h} - x \).

You should use whichever solution you understand best.

**Exercise 1.4**

Change the subject of the formula to the pronumeraal in brackets.

1. \( v = u + at \quad [t] \)
2. \( I = Av^2 + b \quad [A] \)
3. \( A = \pi r^2 \quad [r] \)
4. \( v^2 = u^2 + 2as \quad [u^2] \)
5. \( P = \frac{1}{2} RF^2 \quad [R] \)
6. \( C = \frac{5}{9}(F - 32) \quad [F] \)
7. \( F = \frac{kpq}{r^2} \quad [p] \)
8. \( V = \frac{4}{3} \pi \quad [r] \)
9. \( d = \frac{5}{\sqrt{2}} \quad [h] \)
10. \( s = ut + at^2 \quad [a] \)
You have substituted values into a formula to find a value of the subject. In this section you will extend on this work and look at some problems.

**Rentacar** charges a set fee of \$T per day for the car type and a charge of \$r per kilometre travelled. If a car is hired for \(d\) days and travels \(k\) km, the total hire cost is given by the formula \(C = Td + rk\).

**Easiride** charges a set fee of \$F per day for the car type with unlimited travel. If a car is hired for \(d\) days, the total hire cost is \(C = Fd\).

1. Calculate the cost of hiring a small car from **Rentacar** for 14 days at 80c per km for 2500km and where \(T = \$20\).

2. Calculate the cost of hiring the same car for 14 days from **Easiride** if \(F = \$75\).

**Solutions:**

1. Here \(T = 20\), \(d = 14\), \(r = 0.80\) and \(k = 2500\)

   \[C = Td + rk\]

   \[C = 20 \times 14 + 0.80 \times 2500\]

   \[= 2280\]

   The cost of the car hire is \$2280.

2. Here \(d = 14\) and \(F = 75\)

   \[C = Fd\]

   \[= 75 \times 14\]

   \[= 1050\]

   The cost of the car hire is \$1050.

The difference in the hire costs for the two companies above is quite large for the 14 days. Show that **Rentacar** is cheaper if you travel less than 68km per day over the 14 days.
Exercise 1.5a

1 The numbers of vertices $V$, faces $F$ and edges $E$ of a solid with straight edges and flat faces are related by the formula $E = V + F - 2$.

a Find the number of edges $E$ of a rectangular prism with 8 vertices and 6 faces.

b Find the number of edges of a solid with 6 faces and 6 vertices.

c A crystal has 12 faces and 10 vertices. Find the number of edges.

2 The effective percentage rate of interest on a loan is $E = \frac{2NR}{N+1}$, where $N$ is the number of repayments and $R\%$ is the flat interest rate. Calculate the effective interest rate on a loan which has monthly repayments over 5 years at 10% pa flat interest.

3 The formula $C = A + 2MR$ calculates the cost $C$ of airfreighting a parcel of mass $M$ kg where the fee is $A$ plus $R$ per kg. What is the cost of airfreighting a 5kg parcel at $2.50 per kg and a fee of $10.

4a How long will I take to travel 60 km at a speed of 20 km/h?

b How long will I take to travel $y$ km at a speed of $s$ km/h?

c If a journey consists of travelling a distance of $C$ km through towns at an average speed of $s$ km/h and $M$ km along motorways at an average speed of $v$ km/h, show that the total time taken for the journey is given by $T = \frac{C}{s} + \frac{M}{v}$.

d On a journey from Sydney to Nhill, Rona and Piers travelled 1045 km on motorways at an average speed of 100 km/h and 90 km through towns at an average speed of 40 km/h.

i Calculate the actual driving time.

ii On the journey they take breaks totalling 3 hours. What is the total time taken for the journey?

iii If they left at 4:30 am, what time would they arrive at their destination?
Harder substitutions

These examples involve substituting values and then rearranging to find the value of a pronumeral that is not the subject.

If \( s = ut + \frac{1}{2}at^2 \) find the value of \( u \) when \( s = 54, \ t = 2, \ a = 9.8 \)

Solution:

\[
\begin{align*}
  s &= ut + \frac{1}{2}at^2 \\
  54 &= u \times 2 + \frac{1}{2} \times 9.8 \times 2^2 \\
  54 &= 2u + 19.6 \\
  2u + 19.6 &= 54 \\
  2u &= 54 - 19.6 \\
  2u &= 34.4 \\
  u &= 17.2
\end{align*}
\]

The area \( A \) of a circle of radius \( r \) units is given by \( A = \pi r^2 \). Find, correct to the nearest millimetre, the radius of a circle whose area is 10cm\(^2\).

Solution:

\[
\begin{align*}
  A &= \pi r^2 \\
  10 &= \pi r^2 \\
  r^2 &= \frac{10}{\pi} \\
  r^2 &= 3.183... \\
  r &= \sqrt{3.183...} \\
  &= 1.784... \text{ cm} \\
  &= 17.84... \text{ mm}
\end{align*}
\]

The radius is 18mm.
The frequency of oscillation of a pendulum is \( f = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \) where \( l \) is the length of the pendulum and \( g = 9.8 \). Find the length of a pendulum whose oscillation frequency is 0.2.

**Solution:**

\[
\begin{align*}
\frac{1}{2\pi} \sqrt{\frac{g}{l}} &= f \\
0.2 &= \frac{1}{2\pi} \sqrt{\frac{9.8}{l}} \\
0.4\pi &= \sqrt{\frac{9.8}{l}} \\
0.16\pi^2 &= \frac{9.8}{l} \\
0.16\pi^2 \times l &= 9.8 \\
l &= \frac{9.8}{0.16\pi^2} \\
&= 6.2059...
\end{align*}
\]

The length of the pendulum is 6.2m.

The last calculator step in the example above is

\[
9.8 \div (0.16 \times \pi \times \pi^2) \equiv 6.2059225
\]

If you did not get this answer, it could be because you did not keep the \( 0.16 \pi^2 \) together in the denominator. Make sure you can do this calculation.
Exercise 1.5b

1. If $I = \frac{PRN}{100}$, find $R$ if $P = 500$, $N = 8$ and $I = 360$.

2. The energy of a moving body can be calculated from $E = \frac{1}{2}mv^2$
   where $E$ is the energy, $m$ is the mass and $v$ is the velocity.
   a. What is the mass of an object with velocity 20 m/s and energy 100 joules?
   b. Find the velocity in m/s, if the energy spent is 23.8 joules and the mass is 9.2 kg.

3. The formula connecting the height, $h$ m, of an object above sea level and its distance, $D$ km, to the visible horizon is $D = \sqrt{\frac{25h}{2}}$.
   How high above sea level is a person looking at the horizon 16 km away?

4. The formula $C = \frac{5}{9}(F - 32)$ gives the relationship between degrees Celsius and degrees Fahrenheit. Use it to find the equivalent temperature in degrees Fahrenheit at:
   a. 0°C
   b. 27°C
   c. -20°C

5. The effective percentage rate of interest is $E = \frac{2Rn}{n+1}$ where $R\%$ is the flat rate interest and $n$ is the number of periods. Find the flat rate of interest when $E = 25\%$ and $n = 50$.

6. Einstein’s formula $E = mc^2$ links a body’s energy, $E$ joules, its mass, $m$ kg, and the speed of light $c = 3\times 10^8$ m/s. Find:
   a. the energy released by a 2.5kg body. Give answer in scientific notation, correct to three significant figures.
   b. the mass needed to produce $1.45\times 10^{11}$ joules of energy. Give answer in scientific notation, correct to two significant figures.
Mixed review exercise

You can choose to do this exercise now or use as revision later.

### Exercise 1.5c

1. The depreciation formula \( A = P \left(1 - r\right)^n \) gives the value, $A$, of an item after $n$ years where the initial value is $P$ and the annual depreciation rate as a decimal is $r$.
   
   a. Calculate the value of a car after 5 years if its initial cost was $25,000 and the annual depreciation rate 22% pa. Give your answer to the nearest dollar.
   
   b. A TV set is worth $300 after 4 years depreciating at 10% pa. What was its initial cost? Answer to the nearest $10.

2. Hima completes a 42.2km marathon in 2 hours and 45 minutes.
   
   Use the formula \( S = \frac{d}{t} \) to calculate her speed in km/h, giving your answer correct to one decimal place.

3. An insect population $P$ can be predicted using the formula $P = 1000 \times 2.718^{0.02t}$ where $P$ is the number of insects and $t$ is the time in days. Use this formula to calculate $P$ when $t = 8$.

4. A person’s body mass index is \( I = \frac{m}{h^2} \) where $m$ kg is the mass and $h$ m is the height. For a healthy person $I$ is between 20 and 25. Angela weighs 70 kg and is 1.72 m tall.
   
   a. Calculate Angela’s body mass index correct to 1 decimal place.
   
   b. Does Angela’s value lie in the healthy range?
   
   c. What is the most Angela can weigh to stay in the healthy range?
The maximum velocity of water waves in m/s is given by \( v = \sqrt{gD} \)
where \( g = 9.8 \text{ m/s}^2 \) is the acceleration due to gravity and \( D \) is the depth of water in metres. Using this formula, a table showing the velocities for different depths has been drawn up.

<table>
<thead>
<tr>
<th>Depth, D m</th>
<th>Velocity, v m/s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>10</td>
<td>9.9</td>
</tr>
<tr>
<td>20</td>
<td>14.0</td>
</tr>
<tr>
<td>30</td>
<td>17.1</td>
</tr>
<tr>
<td>40</td>
<td>19.8</td>
</tr>
<tr>
<td>50</td>
<td>22.1</td>
</tr>
<tr>
<td>60</td>
<td>24.2</td>
</tr>
<tr>
<td>70</td>
<td>26.2</td>
</tr>
<tr>
<td>80</td>
<td>28.0</td>
</tr>
<tr>
<td>90</td>
<td>29.7</td>
</tr>
<tr>
<td>100</td>
<td>31.3</td>
</tr>
<tr>
<td>110</td>
<td>32.8</td>
</tr>
<tr>
<td>120</td>
<td>34.3</td>
</tr>
<tr>
<td>130</td>
<td>35.7</td>
</tr>
<tr>
<td>140</td>
<td>37.0</td>
</tr>
<tr>
<td>150</td>
<td>38.3</td>
</tr>
<tr>
<td>160</td>
<td>39.6</td>
</tr>
<tr>
<td>170</td>
<td>40.8</td>
</tr>
<tr>
<td>180</td>
<td>42.0</td>
</tr>
<tr>
<td>190</td>
<td>43.2</td>
</tr>
<tr>
<td>200</td>
<td>44.3</td>
</tr>
<tr>
<td>210</td>
<td>45.4</td>
</tr>
</tbody>
</table>
Use the table to find:

1. the average velocity of waves in a lake of depth 30m
2. the average velocity of waves in a bay of depth 165m
3. the average depth of water in a basin if the fastest average velocity of waves in the basin is 30 m/s

Solutions:

1. When \( D = 30 \), \( v = 17.1 \) m/s

2. 

   \[
   \text{Velocity at 165cm} = \frac{\text{velocity at 160cm} + \text{velocity at 170cm}}{2} \\
   = \frac{39.6 + 40.8}{2} \\
   = 40.2 \text{ m/s}
   \]

3. About 92m. Since 30 m/s lies between 29.7 m/s (depth 90m) and 31.3 m/s (depth 100 m), you need to estimate the depth.

Here is a graph of the relationship \( v = \sqrt{gD} \) which gives the maximum velocity of water waves.

1. Use the graph to find the velocity of waves in an estuary with an average depth of 70m.

2. What might the depth of water be where waves travel at 37 m/s?

3. True or false. Doubling the depth doubles the velocity of the waves.
4 What happens to the velocity of an ocean wave as it approaches the shore?
5 Estimate the greatest velocity of ocean waves in a region where the depth is 300m. Comment on this estimate.

Solutions:
1 About 26.5 m/s
2 About 138m
3 False. For example, the velocity at 50m is 22m/s, at 100m is 31.5m/s and at 200m is 44m/s.
4 It slows down. The depth at the shore is 0m.
5 About 55 m/s. You need to extend the curve to find this estimate.

Note: While tables and graphs have their advantages, they can’t cover all situations. You may need to interpolate (estimate between two values) or extrapolate (extend the curve to find a value beyond the given range).

1 Give one advantage of presenting information as a table.
2 Give one advantage of displaying information in a graph.

Solutions:
1 A table is easy to read and allows you to obtain values at a glance.
2 You can see immediate trends in the data as well as obtain a range of values. Graphs are also visually pleasant to look at.
Exercise 1.6a

1. The distance, $d$ m that an object falls in $t$ s can be approximated by the formula $d = 5t^2$. Create a table of values for this function for $t = 0$ to $t = 10$ with one second increments.

Use the table to answer the following questions:

a. How far does an object fall in the first six seconds?

b. A stone dropped from the top of a cliff takes 5 seconds to reach the bottom. How high is the cliff?

c. How far would an object fall between:
   i. the first and second second
   ii. the seventh and ninth second

d. How long does it take an object to fall 300m?

2. The following table gives the volume of a sphere, correct to the nearest whole number, for different radii.

<table>
<thead>
<tr>
<th>Radius, cm</th>
<th>Volume, cm³</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>34</td>
</tr>
<tr>
<td>3</td>
<td>113</td>
</tr>
<tr>
<td>4</td>
<td>268</td>
</tr>
<tr>
<td>5</td>
<td>524</td>
</tr>
<tr>
<td>6</td>
<td>905</td>
</tr>
<tr>
<td>7</td>
<td>1437</td>
</tr>
<tr>
<td>8</td>
<td>2145</td>
</tr>
<tr>
<td>9</td>
<td>3054</td>
</tr>
<tr>
<td>10</td>
<td>4189</td>
</tr>
<tr>
<td>11</td>
<td>5575</td>
</tr>
<tr>
<td>12</td>
<td>7238</td>
</tr>
<tr>
<td>13</td>
<td>9203</td>
</tr>
<tr>
<td>14</td>
<td>11,494</td>
</tr>
<tr>
<td>15</td>
<td>14,137</td>
</tr>
<tr>
<td>16</td>
<td>17,157</td>
</tr>
<tr>
<td>17</td>
<td>20,580</td>
</tr>
<tr>
<td>18</td>
<td>24,429</td>
</tr>
<tr>
<td>19</td>
<td>28,731</td>
</tr>
<tr>
<td>20</td>
<td>33,510</td>
</tr>
</tbody>
</table>

a. What is the volume of a sphere having radius
   i. 6cm
   ii. 17.5cm
b A sphere has volume $5575\text{cm}^3$. What is its radius?

c If the radius is doubled, by how many times does the volume increase? [Hint: choose any radius from the table and double it.]

d A sphere has volume $1650\text{cm}^2$.
   i Estimate its radius.
   ii Use the formula $V = \frac{4}{3}\pi r^3$ to find the radius of this sphere correct to one decimal place. How does your calculated answer compare with your estimate in part i?

3 For a set of uniform waves, the formula $L = 1.56T^2$ gives the wavelength $L$ m in terms of the time $T$ s for two successive waves to pass a given point.
   a Draw up a table of values of $T$ from 1 to 5 seconds.
   b The graph in the Appendix is drawn for this curve for values of $T$ up to 14 seconds. From the graph find:
      i the wavelength if $T = 7.5\text{s}$?
      ii the time it takes two successive waves to pass if $L = 200\text{m}$.
   c The speed, $v$ m/s of a series of waves is $v = \frac{L}{T}$. Calculate the speed of the waves in part b.
   d In a swell, waves have a wavelength 300m. Find:
      i the time for two successive waves to pass
      ii the speed of the waves in m/s
   e One of the largest recorded set of uniform waves had wavelengths of 760m. Find:
      i the time for two successive waves to pass
      ii the speed of the waves in m/s and km/h
Estimation and Refinement

To solve some equations that arise in practical situations, we use an estimation and refinement technique. This method is sometimes called guess and check.

Solve \(1.05^x = 2\) giving your answer correct to 1 decimal place.

Solution:

First guess, try \(x = 10\)  
\(1.05^{10} = 1.628\ldots\) Too small

Now refine your estimate.

Second guess, try \(x = 15\)  
\(1.05^{15} = 2.078\ldots\) Too big but close

Third guess, try \(x = 14\)  
\(1.05^{14} = 1.979\ldots\) Too small but closer

Fourth guess, try \(x = 14.2\)  
\(1.05^{14.2} = 1.999\ldots\) Bingo!

Hence the answer is \(x = 14.2\).

Note: If you try \(x = 14.3\) you will find that the answer is not as close to 2 as for \(x = 14.2\).

Exercise 1.6b

1. Gayle invested $9000 at 7%pa compounded annually.
   a. Use the formula \(A = P(1 + r)^n\) to show that the amount \(A\) she will have after \(n\) years is given by \(A = 9000(1.07)^n\).
   b. How much will Gayle have after 5 years?
   c. Gayle needs $18 000 for an overseas trip. How many years, correct one decimal place, will it take for the investment to reach this amount? [Hint: Estimate \(n\) then use the guess and check method.]

2. Due to drought, the rabbit population in Wanaring is decreasing by 40% per year. The population is given by \(P = 400(0.6)^n\) where \(n\) is the number of years after 1999. What year will the rabbit population be 100?
Do you understand the following words or expressions?
Look back in your notes, if necessary, and give your explanations here.
pronumeral _______________________________________________
 substitution _______________________________________________
algebraic expression _______________________________________
like term _________________________________________________
changing subject of formula __________________________________
Exercises

1. If \( x = 2, \ y = 5 \) and \( z = 25 \) evaluate the following expressions.
   a. \( 3z - 2y \)
   b. \( xyz \)
   c. \( x^2 + y^2 + z^2 \)
   d. \( \frac{z + x}{y - x} \)
   e. \( \sqrt[3]{\frac{x^2}{z - y}} \)
   f. \( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \)
   g. \( 2y^3 - xz^2 \)
   h. \( \sqrt[3]{z^3} \cdot x \)
   i. \( 5 - \frac{\pi}{xy} \)

2. If \( y = 3x^2 + 2x - 5 \), find \( y \) when \( x = 4 \).

3. a. What is the change in cents from $5 when four articles are purchased for \( m \) cents each?
   b. What is the largest possible value of \( m \)?

4. Simplify:
   a. \( 3z - 2y + 4z \)
   b. \( x(x + 1) - 2x(3 - 2x) \)
   c. \( 3ab + 4a^2 b^2 + 2a^2 b \)
   d. \( \frac{(4y)^2}{2y^2} \)

5. What is the perimeter of a square with side \( (4d - 3) \) cm?

6. Change the subject of the formulas to the pronumeral in brackets.
   a. \( E = mc^2 \) \([c]\)
   b. \( v^2 = u^2 + 2gs \) \([g]\)
7 Solve the following equations.
   a \( 3z - 2 = 22 \)
   b \( 3(x + 1) - 2(3 - 2x) = 9 \)
   c \( \frac{15 + y}{3} = 7 \)
   d \( \frac{8m + 1}{2} + \frac{m}{3} = m + 4 \)
   e \( 3(b - 6) = 2(4 + b) \)
   f \( 2w(w + 5) = 32 + 10w \)

8 The percentage of antifreeze in the mixture in a car’s radiator can be calculated using the formula \( \frac{M - W}{A - W} \times 100 \) where \( M \) is the density of the mixture, \( W \) the density of water and \( A \) the density of the antifreeze. If the mixture has a density of 0.98 and the density of water and antifreeze are 1.0 and 0.8 respectively, calculate the percentage of antifreeze in the mixture.

9 On a trip from Sydney to Brisbane a Boeing 717 uses about 3 tonne of fuel.
   a Calculate the number of kg of fuel used. \([1t = 1000kg]\)
   b The density of a substance is given by \( D = \frac{m}{v} \) where:
      \( D \) = density in kg/L; \( m \) = mass in kg; \( v \) = volume in L.
      Calculate the volume of fuel used on a trip from Sydney to Brisbane if the density of jet fuel is 0.816 kg/L.
   c Calculate the cost of this fuel at 85 cents per litre.
   d Write a formula which allows you to calculate the cost of fuel per passenger.
   e If a Boeing 717 can carry a full passenger load of 117 people, calculate the fuel cost per passenger for the trip.

10 The average density, \( d \) kg/m\(^3\), of the Earth can be calculated using the formula \( d = \frac{3g}{4\pi GR} \) where \( g = 9.8 \) m/s\(^2\) is the acceleration due to gravity; \( G = 6.67 \times 10^{-11} \) is a constant and \( R = 6.37 \times 10^6 \) m is the Earth’s radius. Calculate the average density of the Earth correct to the nearest whole number.
11 Gravity on the moon is much less than that on Earth. The distance, \(d\) m, that an object falls on the moon in \(t\) s can be approximated using the formula \(d = \frac{4}{5}t^2\). Create a table of values for this function for \(t = 0\) to \(t = 10\) with one second increments. Use the table to answer the following questions:

a How far does an object fall in the first five seconds?

b A stone dropped from the top of a cliff takes 7 seconds to reach the bottom. How high is the cliff?

c How far would an object fall between

i the first and second second

ii the eighth and tenth second

d How long does it take the object to fall 50m?

12 Erin made two errors in her solution to the following equation.

\[
\begin{align*}
2(x + 5) - 4(x - 6) &= 27 \\
2x + 10 - 4x - 24 &= 27 \quad \text{(line 1)} \\
-2x - 14 &= 2 \quad \text{(line 2)} \\
-2x &= 16 \quad \text{(line 3)} \\
x &= 20.5 \quad \text{(line 4)} \\
\end{align*}
\]

Which lines do not follow correctly from the previous lines?

A Line 1 and line 3  
B Line 1 and line 4  
C Line 2 and line 3  
D Line 2 and line 4

13 Washing removes 20% of a deep stain at each wash.

a Show that three washes removes about half the stain.

b How many washes are needed to reduce the stain to 5% of the original amount?

14 A photocopier enlarges by 150%. Can a picture be enlarged to twice its original area? Explain how this could be done.
When you have finished this unit of work see if you can do these things.

Tick if you can do them with confidence:

- substitute into and evaluate algebraic expressions - linear, quadratic, cubic, as well as those involving square and cube roots
- add and subtract like terms
- multiply and divide algebraic terms and expressions
- change the subject of equations and formulas involving linear and quadratic terms
- solve equations after substituting values
- solve equations arising from practical situations by estimation and refinement
- use positive and negative powers of ten as part of expressing measurements in scientific notation

Ask for further help with any you feel unsure about.
Please write your questions and any other comments here and overleaf.

_________________________________________________________
_________________________________________________________
_________________________________________________________
_________________________________________________________
_________________________________________________________
_________________________________________________________
Graph for use in Exercise 1.6, question 3.

Wavelength against period for wave train
Exercise 1.1

1  a  3.6  b  -7.2  c  -3.56  
d  0.762  e  0.976  f  1  
g  -56.88  h  -1.629  i  6.691  

2   0.125
3   3
4   28
5   40
6   18.0
7   0.769
8   1.47
9   3.298
10  12.9
11  $1.2 \times 10^3 = 1200$
12  1.34 cm
13  29.37 L

Exercise 1.2a

1  a  6d  b  $6x^2$  c  $3as + 5s$
    d  $3w - 4r - r^2$  e  wer  f  $5zc^2 - z^2c$
    g  0  h  $7t + 5$  i  3
    j  $u^4 + 4u^3$

2  a  $(14a - 2)cm$  b  $(4f + 2)cm$  c  $7d cm$
    d  $(3p + 9)cm$  e  $(2y + 2m + 1)cm$  f  $(6d - 4)cm$
Exercise 1.2b

1. a. $12d^3$  
   b. $9x^4$  
   c. $140a^2s^3$  
   d. $16w^3r^3$  
   e. $90w^2c^2r^2$  
   f. $120z^6c^6$

2. a. $8d$  
   b. $8$  
   c. $3$  
   d. $10r$  
   e. $2w$  
   f. $\frac{5z}{4c}$

3. a. $6d$  
   b. $8x^4 + 1$  
   c. $\frac{28a^2s}{5}$  
   d. $wr + 4w^2$  
   e. $12c^2r^2$  
   f. $\frac{8z}{c}$

   g. $-4r$  
   h. $ab$

4. a. $35d$  
   b. $2 + 2x^2$  
   c. $-5as$  
   d. $6wr$  
   e. $1$  
   f. $\frac{x}{3z}$

   g. $-\frac{3}{10h}$  
   h. $2y^3z^2$

5. a. $19d - 3e$  
   b. $7m^2 - 11m$  
   c. $3 + 2b$  
   d. $2xy$  
   e. $35x + 60$  
   f. $30 - 2y^2$

Note that $30 - 2y^2 = -2y^2 + 30$

6. In $(-3d)^2$ everything inside the parentheses is squared. The result is therefore $9d^2$. But in $-3d^2$ only the d is squared. Read this as $-3 \times d^2$.

7. a. $(12a + 18)cm^2$  
   b. $\frac{1}{2}p(2p - 4)cm^2$  
   c. $\frac{1}{2}\pi r^2cm^2$  
   d. $\frac{1}{2}h(8w + 3) cm^2$

44  
AM3: Algebraic skills and techniques
Exercise 1.3a
1  a = 8  
2  b = 3  
3  c = –8  
4  d = –2  
5  e = 3  
6  f = 3.5  
7  g = –2  
8  h = 17  
9  i = 9  
10 j = 0.2

Exercise 1.3b
1  x = 5  
2  p = 13  
3  d = 8  
4  y = 2  
5  y = 10  
6  m = 6  
7  k = 7  
8  z = 50  
9  c = 5  
10 v = 2.25

Exercise 1.3c
1  a = \frac{3}{7}  
2  w = 12  
3  x = 22  
4  r = 7  
5  u = 20  
6  x = 1
Exercise 1.3d

1 \( x = \pm 2 \)
2 \( x = \pm 3 \)
3 \( x = 147 \)
4 \( x = 4 \)
5 \( m = \pm 1.73 \)
6 \( y = 2.24 \)
7 \( w = 23 \)

In question 6 there is only one answer, +2.24. This is because the square root of a number has to be positive. So the \( y \) on the right hand side of the equation must be positive.

Exercise 1.3e

1 \( x = 2\frac{3}{5} \)
2 \( x = -3 \)
3 \( x = \frac{1}{4} \)
4 \( x = 4 \)
5 \( x = 7\frac{1}{2} \)
6 \( x = 108 \)
7 \( x = \pm 8.94 \)
8 \( x = 11\frac{3}{7} \)
9 \( t = 2 \)
10 \( x = 1\frac{1}{3} \)
11 \( x = 4 \)
12 \( x = 5\frac{2}{3} \)
13 \( x = -25 \)
14 \( x = -36\frac{1}{2} \)
15 \( x = 11 \)
16 \( x = -1 \)
17  \( x = 25 \)
18  \( y = 36 \)

**Exercise 1.4**

1  \( t = \frac{v - u}{a} \)
2  \( A = \frac{1 - b}{b^2} \)
3  \( r = \sqrt[4]{\frac{A}{\pi}} \)
4  \( u^2 = r^2 - 2as \)
5  \( R = \frac{2P}{r^2} \)
6  \( F = \frac{9}{5} C + 32 \)
7  \( p = \frac{Fr^2}{kq} \)
8  \( r = \sqrt[4]{\frac{3V}{4\pi}} \)
9  \( h = \frac{2d^2}{25} \)
10  \( a = \frac{s - ut}{r^2} \)

**Exercise 1.5a**

1  a  \( E = 12 \)  b  \( E = 10 \)  c  \( E = 20 \)
2  19.67%
3  $35
4  a  3 hours  B  \( \frac{y}{s} \) hours
   c  \( C \) km at \( s \) km/h takes \( \frac{C}{s} \) hours; motorway journey takes \( \frac{M}{v} \) hours; then add, to get the total time.
   d  i  12.7 hours = 12 hours 42 minutes
      ii  15 hours 42 min  iii  8:12 pm
Exercise 1.5b
1 \( R = 9 \)
2 a \( m = 0.5 \) kg   b \( v = 2.27 \) m/s
3 \( h = 20.5 \) metres
4 a \( 32 \) °F   b \( 80.6 \) °F   c \( -4 \) °F
5 \( 12.75\% \)
6 a \( 2.25 \times 10^{17} \) joules   b \( 1.6 \times 10^{-6} \) kilograms

Exercise 1.5c
1 a $7218   b $460
2 15.35 km/h
3 1173 insects
4 a 23.7   b yes   c about 74 kg

Exercise 1.6a
1

<table>
<thead>
<tr>
<th>t (sec)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d ) (m)</td>
<td>0</td>
<td>5</td>
<td>20</td>
<td>45</td>
<td>80</td>
<td>125</td>
<td>180</td>
<td>245</td>
<td>320</td>
<td>405</td>
<td>500</td>
</tr>
</tbody>
</table>

a 180 m   b 125 m   c i 15 m
ii 160 m   d about 7.5 sec
2 a i 905 cm\(^3\)   ii about 22 500 cm\(^3\)   b 11 cm
 iii 8 times   d i about 7.5 cm   ii 7.3 cm
3 a

<table>
<thead>
<tr>
<th>T (sec)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L ) (m)</td>
<td>1.56</td>
<td>6.24</td>
<td>14.04</td>
<td>24.96</td>
<td>39</td>
</tr>
</tbody>
</table>

b i about 88 m   ii 11.4 sec   c i 11.7 m/s
ii 17.5 m/s   d i 13.9 sec   ii about 21.6 m/s
c i 22 sec   ii 34.5 m/s; 124 km/h

Exercise 1.6b
1 b $12 622.97   c 10.2 years
2 2002
AM4 Modelling linear & non-linear relationships

Part 1: Lines and parabolas
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>3</td>
</tr>
<tr>
<td>1.1 Linear functions</td>
<td>4</td>
</tr>
<tr>
<td>1.2 Intersecting lines</td>
<td>8</td>
</tr>
<tr>
<td>Interpreting a point of intersection</td>
<td>9</td>
</tr>
<tr>
<td>1.3 More on lines</td>
<td>12</td>
</tr>
<tr>
<td>Breakeven points</td>
<td>12</td>
</tr>
<tr>
<td>1.4 Distance / time graphs</td>
<td>16</td>
</tr>
<tr>
<td>Distance-time graphs</td>
<td>16</td>
</tr>
<tr>
<td>1.5 Graphing quadratic functions</td>
<td>19</td>
</tr>
<tr>
<td>Terminology</td>
<td>23</td>
</tr>
<tr>
<td>Exercises</td>
<td>25</td>
</tr>
<tr>
<td>Student evaluation</td>
<td>29</td>
</tr>
<tr>
<td>Appendix</td>
<td>31</td>
</tr>
<tr>
<td>Answers</td>
<td>47</td>
</tr>
</tbody>
</table>

---

Part 1: Lines and parabolas
This is the first of three parts covering the syllabus topic AM4 Modelling linear and non-linear relationships, in the Algebraic modelling component of the course.

**Specific content outcomes**

By the end of Part 1, you will have been given opportunities to:

- generate tables of values and graph linear functions
- interpret the point of intersection of the graphs of two linear functions
- consider linear graphs from practical contexts
- understand the concept of ‘breakeven’ point
- consider distance/velocity/time graphs and problems
- generate tables of values and graph quadratic functions of the form \( y = ax^2 + bx + c \), where \( x \geq 0 \)
- note that different forms of an expression produce identical graphs

*For students in Distance Education Centres only:*

There is an evaluation page at the back of this part; fill it in when you have finished the work; say how easy/hard/interesting you find this work; ask relevant questions and return your comments to your teacher.
Some of the work covered in this unit may already be familiar to you. If this is the case, you should treat this as revision. You should revise AM2: Modelling linear relationships in the preliminary course if you find any of this work unfamiliar.

A linear function is one whose graph is a straight line.

- The gradient intercept form of a straight line is \( y = mx + b \)
- \( m \) is the gradient (slope) and \( b \) is the \( y \)-intercept.
- The general equation of a straight line is \( ax + by + c = 0 \) where \( a, b, \) and \( c \) are constants
- \( x \) is the independent variable and \( y \) is the dependent variable

1. Draw the graph of the linear function \( y = 2x - 3 \).
2. Write down its gradient and \( y \)-intercept.

Solutions:

1. Draw up a table of values and plot the points on a grid.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-3</td>
<td>-1</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

2. Gradient \( m = 2 \) and \( y \)-intercept \( b = -3 \).
Exercise 1.1a

[Grids are available in the Appendix for you to remove and use.]

1 Graph the following linear functions.
   a \( y = x + 1 \)  b \( y = 2 - x \)
   c \( y = 3x - 4 \)  d \( x + y = 4 \)
   e \( 3x - 2y = 3 \)  f \( x + 2y = 1 \)

2 For a, b, c, d in question 1, write down the gradient and \( y \)-intercept.

3 Match the line with its equation.
   a \( y = x \)  b \( y = -2x - 4 \)
   c \( y = -\frac{1}{2}x \)  d \( y = x + 4 \)
   e \( y = \frac{1}{2}x - 2 \)

4 If the \( x \) coordinate of a point is \(-2\), find the \( y \) coordinate given that the point lies on the line \( y = 2x - 3 \).

5 If the \( y \) coordinate of a point is 5, find its \( x \) coordinate given that the point lies on the line \( y = 4 - 3x \).

6 Which point lies on the line \( y = 2x + 3 \)?
   A \((-1, 2)\)  B \((1, -2)\)  C \((-2, -1)\)  D \((-1, -2)\)

7 a Which of these points satisfy the equation \( 4x - 3y = 1 \)?
   \((0, 1)\) \((1, 0)\) \((1, 1)\) \((2, 3)\) \((-2, -3)\) \((0.25, 0)\)

b Graphically, what does ‘satisfies the equation’ mean?
The gradient and y-intercept of a line

- The gradient of a line is a measure of its steepness. Lines sloping right ( / ) have a positive gradient and those sloping left ( \ ) have a negative gradient.

\[
\text{Gradient} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}
\]

- the y-intercept of a line is the distance along the y-axis from the origin (0, 0) to the point where the line cuts the y-axis.

For each line find:

1. the gradient
2. the y-intercept
3. the equation of the line in the form \( y = mx + b \)

Solution:

1. Choose two points on the line, say (-2, 0) and (0, 2).
   Gradient = \( \frac{\text{rise}}{\text{run}} = \frac{2 - 0}{0 - (-2)} = 1 \)
   2. y-intercept is 2
   3. Equation is \( y = x + 2 \)

Solution:

1. Choose two points on the line, say (0, 0) and (-2, 4).
   Gradient = \( \frac{4 - 0}{-2 - 0} = -2 \)
   2. y-intercept is 0
   3. Equation is \( y = -2x \)
Exercise 1.1b

1. For each of the lines shown, write down:
   a) the gradient
   b) the y-intercept
   c) the equation of the line.

2. Match the line with its equation.
   a) \( y = x + 3 \)
   b) \( y = -x - 4 \)
   c) \( y = \frac{1}{2}x \)
   d) \( y = -2x + 4 \)
1.2 Intersecting lines

1. Draw the graphs of the linear functions $y = 3x - 1$ and $y = 3 - x$
2. What is the point of intersection?
3. What is the significance of the point of intersection?

Solutions:
1. The two lines are drawn on the grid and labelled.
2. The two lines intersect at $(1, 2)$.
3. The point $(1, 2)$ satisfies both equations, that is, it obeys the rule $y = 3 - x$ and the rule $y = 3x - 1$.

Exercise 1.2a

[Grids are available in the Appendix for you to remove and use.]

1. a Graph and label the lines $y = x + 3$ and $y = -\frac{1}{2}x$ on the same grid.
   b Where do these two lines intersect?

2. a Graph the lines $y = x + 1$, $2x + y = 7$ and $x + y - 5 = 0$ on the same grid:
   b Where do the three lines intersect?
3 a Graph the lines \( y = x + 3 \) and \( y = x - 2 \) on the same grid:
b Explain why these lines will never intersect.

4 a Find the equation of line Q.
b Where does Q cut the \( x \)-axis?
c Where does P cut the \( y \)-axis.
d Find the equation of line P.
e Where do P and Q intersect?
f Write the equation of the line which passes through the origin \((0, 0)\) and also the point of intersection of P and Q.

**Interpreting a point of intersection**

Most real life problems involve positive variables and hence we are usually only interested in the part of a line in the first quadrant where both \( x \) and \( y \) are positive. For example, a tradeperson’s cost depends on the time spent doing a job.

Some real life problems will involve drawing two graphs in the positive quadrant then finding and interpreting the point of intersection.
The sum of two positive numbers is 40 and their difference is 6.

1. Find two linear functions to represent this information.
2. Using \( x \) values of 0, 10, 20 and 30, draw the two graphs on a grid and hence find the two numbers.

**Solutions:**

1. Let the numbers be \( x \) and \( y \). Then \( x + y = 40 \) and \( x - y = 6 \)

2. Point of intersection is (23, 17) so the two numbers are \( x = 23 \) and \( y = 17 \).

Two rental companies lend identical chainsaws under the following conditions.

Company A: $5 an hour.

Company B: $15 initial rental charge plus $3.50 per hour.

Three people want to rent a chainsaw for the following time periods: Rick, 7 hours; Dick, 10 hours; and Nick, 12 hours.

Which company should each choose, to get the best price?

**Solution:**

Company A: cost equation is \( C = 5t \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C )</td>
<td>0</td>
<td>5</td>
<td>10</td>
</tr>
</tbody>
</table>

Company B: cost equation is \( C = 15 + 3.5t \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C )</td>
<td>15</td>
<td>18.50</td>
<td>22</td>
</tr>
</tbody>
</table>
The two lines intersect at (10, 50).

For 10 h, the cost of $50 is the same for A and B.
For < 10 h, A charges less than B since $C = 5t$ is below $C = 15 + 3.5t$.
For > 10 h, B has the cheaper price since after that time, it is cheaper to rent from company B.

Rick should choose company A. [A’s cost is $35 and B’s is $39.50].
Dick can rent from either company as the cost of $50 is the same.
Nick should choose company B. [B’s cost is $57 and A’s is $60].

Exercise 1.2b

[Grids are available in the Appendix for you to remove and use.]

1. The sum of two numbers $x$ and $y$ is 50 and their difference is 14.
   a. Explain why $x + y = 50$ and $x - y = 14$.
   b. Graph these two lines on a grid and hence find the two numbers.
      [Use $x$-values 0, 10, 20, 30]

2. The cost of two adult tickets and one child ticket is $16. One adult ticket and three child tickets cost $18.
   a. Letting $a$ be the cost of an adult ticket and $c$ the cost of a child’s ticket. Write two linear equations for the given information.
   b. Draw these two lines on a grid and find the cost of each ticket.

3. The number of items people buy depends on the price. The higher the price, the less is the demand.
   On a particular day the supply and demand for tomatoes is:
   Demand equation (consumer) $p = 4 - 0.2q$
   Supply equation (supplier) $p = 0.1q + 0.7$
   where $q$ represents the quantity of tomatoes in thousands of kilograms and $p$ is the cost per kilogram.
   a. Use $q$ values of 0, 10, 20 and 30, and draw the graphs on a grid.
   b. At what price will supply equal demand?
   c. Suggest what might happen if the price of tomatoes is greater than this equilibrium price.
Breakeven points

For a business to receive a profit, the income (money it takes in) must be greater than the costs (money it pays out).

Expenses will occur in any business whether goods are sold or not as some expenses are fixed. Examples include rent, wages and council rates.

Other expenses are variable. Examples include cost of materials, overtime and machinery wear and tear.

For a business we can generate two equations: one for income and the other for costs. Where these two lines meet is called the breakeven point. This is where a business ‘breaks even’, that is, where income equals costs. Any more sales will result in a profit for the company.
For a CD manufacturer: cost equation is $C = 300 + 1.5x$ and income equation is $I = 2x$ where $x$ is the number of CD’s sold in a week. How many CD’s must be sold to make a profit?

**Solution:**

Firstly, graph the two linear functions.

\[
C = 300 + 1.5x
\]

\[
\begin{array}{c|c|c|c}
  x & 0 & 200 & 400 \\
  c & 300 & 600 & 900 \\
\end{array}
\]

\[
I = 2x
\]

\[
\begin{array}{c|c|c|c}
  x & 0 & 200 & 400 \\
  l & 0 & 400 & 800 \\
\end{array}
\]

The breakeven point here is (600, 1200). So 600 CD’s need to be sold to generate an income of $1200 to exactly match the cost involved.

**Note:** if more than 600 CD’s are sold, this results in a profit for the company as the ‘income’ line will be above the ‘cost’ line.
Use the graph in the example above to answer these questions:

1. Will sales of 200 CD’s in a week result in a profit or a loss? Explain.

2. What is the greatest loss the company can sustain in a week? When might it occur?

3. How many CD’s must be sold to make a profit of $100 per week?

Solutions:

1. From the graph, the costs are $600 but the income is only $400 so there is a $200 loss.

2. The greatest loss is when no CD’s are sold but fixed costs of $300.

3. Look for a difference of 100 in the 2 lines after the breakeven point. This occurs when \( x = 800 \). [Hence \( C = 1500 \) and \( I = 1600 \)].

Exercise 1.3

[Grids are available in the Appendix for you to remove and use].

1. A factory produces computer disks and sells them for 50¢ each. It has fixed costs of $2100 per week plus variable costs of 20¢ for each disk manufactured.
   a. If \( x \) is the number of disks made per week, show that costs are given by the equation \( C = 2100 + 0.2x \) and income by \( I = 0.5x \).
   b. Using \( x \) values from 0 to 10 000, graph and label the two lines.
   c. How many disks must be produced each week to break even?
   d. How much profit does the factory make when it produces 9000 disks per week?
   e. In a particular week staff went on strike and only 3000 disks were produced. Did the factory make a profit or loss? How much was this?
2 Fixed costs for a sunglasses manufacturer are $9000 per week with variable costs of $3.50 per pair of sunglasses.

a Why is the weekly cost \( C = 9000 + 3.5x \), where \( x \) is the number of sunglasses produced each week?

b These glasses are sold to retailers at $5 each. Show that the weekly income from these sales is \( I = 5x \).

c On a grid draw two lines and label them. Make the horizontal axis 0 to 10 000 glasses and the vertical axis from 0 to $50 000.

d How many sunglasses must the factory produce to break even?

e If the manufacture wanted to make a weekly profit of $10 000, how many sunglasses must be produced?

3 A watch manufacturer recorded the income and costs involved in making the watches.

<table>
<thead>
<tr>
<th>Sales (watches per week)</th>
<th>Total costs ($)</th>
<th>Income ($)</th>
<th>Financial outcome ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50 000</td>
<td></td>
<td>50 000 loss</td>
</tr>
<tr>
<td>300</td>
<td>60 000</td>
<td></td>
<td>30 000 loss</td>
</tr>
<tr>
<td>600</td>
<td>70 000</td>
<td>60 000</td>
<td>10 000 loss</td>
</tr>
<tr>
<td>900</td>
<td>80 000</td>
<td></td>
<td>10 000 profit</td>
</tr>
<tr>
<td>1200</td>
<td>90 000</td>
<td></td>
<td>30 000 profit</td>
</tr>
</tbody>
</table>

a Explain why the income was $60 000 when 600 watches were produced.

b Complete the income column.

c Plot the values on a grid and draw a cost line and an income line.

d i What are the fixed costs?

d ii How is this shown on the graph?

e How many watches must be produced before a profit is made?

f Find the profit in a week when 1500 watches are produced.
Distance – time graphs

Janet lives on a farm several kilometers from the nearest town. The graph shows Janet’s trip to town and back on a particular day.

1. When did Janet start her journey?

2. How far from home was she at:
   a. 10:30 am    b. 4:15 pm

3. When did she arrive in town and how long did she spend there?

4. On her way to town she stopped to chat with a friend for half an hour, between what times was this?

5. a. What was the total distance traveled?
b How long did the trip take?
c What was Janet’s average speed for the trip?

**Solutions:**

1  9 am

2  a 120 km  b  60 km

3 She arrived at noon and stayed 2 \( \frac{1}{2} \) hours.

4 From 11:00 – 11:30 am

5  a 400 km  b 8 hours

c  

\[
\text{Speed} = \frac{\text{distance covered}}{\text{time taken}}
\]

\[
= \frac{400}{8}
\]

\[
= 50 \text{ km/h}
\]

**Note:** This is the speed for the entire journey and includes the times she was stationary.

**Exercise 1.4**

1 The following graphs shows Arthur’s movement from his home to his local shop and back.
Use this graph to answer the following questions:

a  How long did it take Arthur to walk to the shop?
b  How far from home is the shop?
c  How long did he spend at the shop?
d  How far from home was he after 3 minutes?
e  Find his average speed:
   i  on his way to the shop
   ii on his way home from the shop
   iii for the entire trip.

2  The following graph shows the motion of a particle over time.

   a  How far from the origin was the particle initially?
   b  What distance did the particle cover in the first four seconds?
   c  How far was the particle from its starting point after eight seconds?
   e  Between what times was the particle stationary?
   f  Find the particle’s average speed
      i  between 0 and 2 seconds
      ii between 2 and \(\frac{3}{2}\) seconds.
   g  Between which two times was the particle traveling the fastest?
      How do you know?
Quadratic functions

In this section we will consider the quadratic function \( y = ax^2 + bx + c \).

- The graph of a quadratic function is a **parabola**.

- Each parabola has an axis of symmetry (a line dividing the parabola into two halves) and a maximum or minimum turning point.

- A parabola with a **minimum** turning point is **concave up**.

- A parabola with a **maximum** turning point is **concave down**.

The cross-sections of radio-telescope dishes is the shape of a **parabola**.

In a parabola reflector such as the headlight of a car, light from a light source is bounced off the reflector and emerges as a parallel beam of light.
Graph the parabola \( y = x^2 \) for values of \( x \) between 0 and 4.

**Solution:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
</tr>
</tbody>
</table>

Draw up a table of values

Plot these values and join them to get a smooth curve.

*Note:* This curve is only half of the parabola \( y = x^2 \). The other half is in the second quadrant. The curve is **concave up** because it has minimum turning point. This can also be determined because the coefficient of \( x^2 \) is positive.
1. Complete the table of values for \( y = 2x^2 - 8x + 5 \) then sketch the parabola.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y  )</td>
<td>5</td>
<td>-1</td>
<td>-3</td>
<td>-1</td>
<td>5</td>
</tr>
</tbody>
</table>

2. What is the minimum turning point?

3. What is the equation of the axis of symmetry?

**Solutions:**

2. The minimum turning point is \((2, -3)\).

3. The axis of symmetry is \( x = 2 \).

**Note:** The axis of symmetry divided the parabola into two halves. This is concave up.
Exercise 1.5

1 Which of the following are equations of parabola?

a \( y = x - 9 \)  
b \( y = 2 - x^2 \)

c \( y = x^2 - 4x + 1 \)  
d \( y = 4 - x^2 \)

e \( y = x(x - 3) \)  
f \( y = 2x - 4 \)

[Grids are available in the Appendix for you to remove and use.]

2 On the same grid using values of \( x \) from 0 to 5, draw and label the following parabolas.

a \( y = x^2 - 6 \)  
b \( y = 10 - x^2 \)

c \( y = 2x^2 - 6x \)  
d \( y = x^2 - 4x + 1 \)

3 On the same grid, with values of \( x \) from 0 to 4, draw the graphs of \( y = x^2 + x \) and \( y = -x^2 + x \). Label your curves.

a Are the curves concave up or down?

b What tells you which way a curve faces?

4 Without drawing them, which of the following parabolas are concave up?

a \( y = 2x^2 - 9x + 7 \)  
b \( y = 10x - x^2 + 3 \)

c \( y = x^2 - 6x + 5 \)  
d \( y = 5x^2 - 4x + 8 \)

e \( y = 3 - 2x + x^2 \)  
f \( y = \frac{1}{2}x^2 \)

g \( y = x - 6x^2 + 5 \)  
h \( y = -5x^2 - 4x + 3 \)

5 Draw these parabolas on the same grid for values of \( x \) from 0 to 3.

\[ y = \frac{1}{2}x^2 \]  \[ y = x^2 \]  \[ y = 2x^2 \]

Comment on the differences between each parabola.
Do you understand the following words or expressions?
Look back in your notes, if necessary, and give your explanations here.

linear function _____________________________________________
________________________________________________________________
coordinates _________________________________________________
________________________________________________________________
ordered pair _________________________________________________
________________________________________________________________
gradient ____________________________________________________
________________________________________________________________
y-intercept __________________________________________________
________________________________________________________________
point of intersection __________________________________________
________________________________________________________________
breakeven point ______________________________________________
________________________________________________________________
fixed cost ____________________________________________________
________________________________________________________________
variable cost _________________________________________________
________________________________________________________________
quadratic function ____________________________________________
________________________________________________________________
parabola

axis of symmetry

turning point
1. Graph the lines \( y = x - 3 \) and \( 2x + y = 3 \) on the same grid for \( x \) values from -5 to +5.
   a. What is the gradient and \( y \)-intercept of each line?
   b. What is the point of intersection?

2. The sum of two numbers is 12. When one number is subtracted from twice the other the result is 3.
   a. If the two numbers are \( x \) and \( y \), explain why \( x + y = 12 \) and \( 2x - y = 3 \).
   b. Graph and label these two lines on a grid.
   c. What are the values of \( x \) and \( y \)?

3. A rectangle with length \( l \) and width \( w \) has perimeter 40cm. If 2cm is taken from its length and added to its width, the rectangle becomes a square.
   a. Write two linear equations showing this information
   b. Graph these two linear equations on the same grid
   c. What are the dimensions of the rectangle?

4. The number of chirps per minute, \( n \), that a cricket makes at temperature \( T \) °C can be approximated by the formula \( n = 87 - 40T \).
   a. On a grid, graph this line for values of \( T \) from 10 to 50.
   b. Comment on the relationship between the temperature and the number of cricket chirps per minute.
   c. What is the number of chirps per minute at 32°C.
   d. Explain why this relationship does not hold below temperatures of about 5°C.
5 Norm’s tarpaulin factory has costs and expenses which closely follow these two equations: \[ C = 3000 + 1.5x \] and \[ I = 2.5x \] where \( C \), \( I \) and \( x \) represent the weekly costs, income and tarps produced.

a. Graph the two linear equations on a grid.

b. How many tarps must be produced per week for the company to make a profit?

c. What is the weekly fixed cost for Norm’s tarpaulin factory?

6 The following graph is a distance time graph for Monica’s journey.

a. When did Monica begin her journey?

b. At what times did she stop to rest?

c. Find her average speed between:
   i. 9am and 10am
   ii. 12:30pm and 2:30pm

d. Between which two times was Monica travelling the fastest?

e. At 5:30pm she began travelling towards home at an average speed of 60 km/h. How far from home was she at 7pm?

f. Sunset on this day was at 7:10pm. How far from home was Monica at this time?
What is the equation of the line of best fit as shown?

A $y = \frac{x}{4} + 3$
B $y = \frac{x}{3} + 4$
C $y = 3x - 12$
D $y = 3x + 4$
When you have finished this unit of work see if you can do these things.

<table>
<thead>
<tr>
<th>can do with confidence</th>
<th>need more help</th>
</tr>
</thead>
</table>

**Tick if you can do them with confidence:**

- generate tables of values and graph linear functions
- interpret the point of intersection of the graphs of two linear functions
- consider linear graphs from practical contexts
- understand the concept of ‘breakeven’ point
- consider distance/velocity/time graphs and problems
- generate tables of values and graph quadratic functions of the form $y = ax^2 + bx + c$, where $x \geq 0$
- note that different forms of an expression produce identical graphs

Ask for further help with any you feel unsure about.
Please write your questions and any other comments here and overleaf.

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________
These grids are for use in Exercise 1.1a and 1.2a.
AM 4: Modelling linear and non-linear relationships
These grids are for use in Exercise 1.2b

![Graphs for Exercise 1.2b](image)
These grids are for use in Exercise 1.3 and 1.4.
Exercise 1.5 Question 2
Exercise 1.5 Question 3
Exercise 1.5 Question 5
End of activity Exercise. Question 2
Question 3

Question 4
Question 5
Exercise 1.1a

1. a

2. a \( m=1, b=1 \)  b \( m=-1, b=2 \)
   c \( m=3, b=-4 \)  d \( m=-1, b=4 \)
   e \( m=\frac{3}{2}, b=-\frac{3}{2} \)  f \( m=-\frac{1}{2}, b=\frac{1}{2} \)

3. a \( R \)  b \( P \)  c \( T \)  d \( Q \)  e \( S \)

4. \( y=-7 \)
5 \( x = \frac{1}{3} \)

6 C

7 a (1, 1) (–2, –3) (0.25, 0) b the point lies on the line

Exercise 1.1b

1 line H: gradient = –1; y-intercept = -3; equation, \( y = -x - 3 \)
line I: gradient = -2; y-intercept = 2; equation, \( y = -2x + 2 \)
line J: gradient = 2; y-intercept = 0; equation, \( y = 2x \)
line K: gradient = \( \frac{1}{2} \); y-intercept = 2; equation, \( y = \frac{1}{2}x + 2 \)

2 a X b W c Y d Z

Exercise 1.2a

1 a

b (–2, 1)

3 a

B Lines are parallel. In fact, all lines which have the same gradient are parallel.

4 a \( y = -\frac{1}{2}x + 4 \) b \( x = 8 \)
   c \( y = -1 \) d \( y = 2x - 1 \)
   e (2, 3) f \( y = \frac{3}{2}x \)
Exercise 1.2b

1  a  If the two positive number are $x$ and $y$, their sum is $x + y$ and this equals 50. If $x$ is bigger than $y$, their difference is $x - y$ which has a value of 14.

\[\begin{align*}
\text{Graph:} & \quad x = 32, \ y = 18
\end{align*}\]

2  a  $2a + c = 16$ and $a + 3c = 18$.

\[\begin{align*}
\text{Graph:} & \quad \text{An adult ticket costs } $6, \text{ while a child’s ticket costs } $4.
\end{align*}\]

3  a  \[p = 4 - 0.2q\]

\[
\begin{array}{|c|c|c|}
\hline
q & 0 & 10 & 20 \\
\hline
p & 4 & 2 & 0 \\
\hline
\end{array}
\]

\[\begin{align*}
\text{Graph:} & \quad $1.80 per kilogram \\
\text{b} & \quad \text{Supply would be greater than demand, and some tomatoes would not be sold.}
\end{align*}\]
**Exercise 1.3**

1. **a** Cost is fixed cost + variable cost. The fixed cost is $2100 per week, while the variable cost is $0.20 for each item sold. With $x$ items sold each week, the cost is $C = 2100 + 0.2x$. The income produced is $0.50 for each item, hence $I = 0.5x$.

   ![Graph](image1.png)

   **b**

   **c** 7000
   **d** About $600
   **e** Loss of about $1200

2. **a** On top of the fixed costs, it costs him $3.50 for each pair of sunglasses produced.

   **b** Income = selling price × number of sunglasses sold each week

   ![Graph](image2.png)

   **c**

   **d** 600
   **e** About 14 000 sunglasses. (You need to extend both graphs until the vertical displacement between them is $10 000$.)

3. **a** It cost $70 000 to produce these watches but the manufacturer lost $10 000 so the income must have been $60 000.

   Profit/loss = Income - Costs.

   ![Graph](image3.png)

   **b** 0, 30 000, 90 000, 120 000
   **c**

   **d** i $50 000
   **ii** y-intercept (ie when number of watches is zero)

   **e** About 750
   **f** $50 000
Exercise 1.4

1  a  4 minutes  b  300 m  c  5 minutes  d  225 m  
   e  i  75 m/min  ii  150 m/min  iii  $\frac{600}{11}$ m/min  

2  a  5 cm  b  15 cm  c  12.5 cm  
   d  $25 + 12.5 = 37.5$ m  
   e  Between 3.5 and 4.5 seconds, and between 5 and 6.5 seconds.  
   f  i  2.5 cm/s  ii  6.7 cm/s  
   g  4.5 and 5 seconds. Slope of line steepest.

Exercise 1.5

1  c, d, e are parabolas as they are of the form $y = ax^2 + bx + c$.  

2  

![Graph showing parabolas]

Part 1: Lines and parabolas 51
3  a  

$y = x^2 + x$ is concave up; $y = -x^2 + x$ is concave up.

b  When the coefficient of the $x^2$ term is positive, the parabola is concave up. When the coefficient of the $x^2$ term is negative, the parabola is concave down.

4  a  concave up  b  concave down  
c  concave up  d  concave up  
e  concave up  f  concave up  
g  concave down  h  concave down

5  (Graphs not drawn.) As the coefficient of the $x^2$ term becomes larger, the parabola becomes steeper.
AM4 Modelling linear & non-linear relationships

Part 2: Non - Linear graphs
## Contents

- Introduction ........................................................................................................... 2
- 2.1 Practical parabolas ......................................................................................... 3
- 2.2 Cubic equations ............................................................................................. 8
- 2.3 Hyperbolas ..................................................................................................... 10
- 2.4 Exponential curves ......................................................................................... 14
- 2.5 Miscellaneous problems ................................................................................ 17
- Terminology .......................................................................................................... 20
- Exercises .............................................................................................................. 23
- Student evaluation ............................................................................................... 27
- Appendix ............................................................................................................... 29
- Answers ............................................................................................................... 89
Introduction

This is the second of three parts covering the syllabus topic AM4: Modelling linear and non-linear relationships, in the Algebraic modelling component of the course.

Specific content outcomes

By the end of Part 2, you will have been given opportunities to:

- use the quadratic graph to find minimum and maximum values in practical contexts
- generate tables of values to graph cubic, exponential and hyperbolic functions using pen and paper
- recognise that for $a > 1$, $y = b(a^x)$ represents exponential growth and for $0 < a < 1$, it represents exponential decay
- use algebraic functions as models of physical phenomena
- recognise the limits of models when interpolating and/or extrapolating

For students in Distance Education Centres only:

There is an evaluation page at the back of this part; fill it in when you have finished the work; say how easy/hard/interesting you find this work; ask relevant questions and return your comments to your teacher.
In a previous unit of work you were introduced to parabolas. A parabola is the graph of the quadratic function \( y = ax^2 + bx + c \). Parabolas are found in everyday life, for example, the path a ball follows when hit or thrown is a parabola and a car’s headlight is parabolic.

Basil throws a ball from the top of a building. The height, \( h \) m, of the ball above the ground is given by the formula \( h = 22 + 21t - 5t^2 \) where \( t \) is the time measured in seconds.

1. Complete the table for this equation.

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h ) (m)</td>
<td>38</td>
<td></td>
<td></td>
<td></td>
<td>26</td>
<td></td>
</tr>
</tbody>
</table>
2 Plot these points and draw a smooth curve through them.

3 What is the height of the building?

4 a How high above the building did the ball reach?
   b When was this?

5 Estimate when the ball will hit the ground.

6 For how long was the ball above the height of the building?

Solutions:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>t (s)</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>h (m)</td>
<td>22</td>
<td>38</td>
<td>44</td>
<td>40</td>
<td>26</td>
</tr>
</tbody>
</table>

![Graph showing the height of the ball over time]
3 The building is 22m high. [when \( t = 0, h = 22 \text{m} \)]

4 \( a \) The ball reached a maximum height above ground level of 44m which is 22 m above the building.

\( b \) After 2 seconds.

5 At about 5.1seconds. [Continue the graph until it meets the \( t \)-axis ]

6 The ball is above the height of the building for 4.2s. [from 0 to 4.2s]

**Note**: In a realistic graph, only positive values of \( t \) and \( h \) have meaning.

### Exercise 2.1

1 An arrow is fired into the air and its height above the ground after \( t \) seconds is \( h = 5(10 - t^2) \).

\( a \) Draw up a table of values from \( t = 0 \) to \( t = 10 \).

\( b \) Plot these points on the grid in the Appendix and draw a smooth curve.

\( c \) Give the graph a suitable title.

\( d \) What is the time of flight for the arrow?

\( e \) What maximum height did the arrow reach?

2 How far you can see to the horizon depends on how far above sea level you are. The formula \( h = \frac{2}{25}d^2 \) gives the height \( h \) m above sea level in terms of the distance \( d \) km you can see to the horizon.
a Use this formula to find values of \( h \) for values of \( d \) from 0 to 100.

b Plot these points on a grid and draw a smooth curve through them.

c How far can you see to the horizon from 145m above sea level?

d If the horizon is 85km away, what is your height above sea level?

3 A new car was test-driven at various speeds and its petrol consumption recorded.

```
<table>
<thead>
<tr>
<th>Speed (km/h)</th>
<th>Petrol consumption (litres/100km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>30</td>
<td>6</td>
</tr>
<tr>
<td>40</td>
<td>8</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
</tr>
<tr>
<td>60</td>
<td>12</td>
</tr>
<tr>
<td>70</td>
<td>14</td>
</tr>
<tr>
<td>80</td>
<td>16</td>
</tr>
</tbody>
</table>
```

a What petrol consumption was recorded at 70 km/h?

b Estimate the petrol consumption at 35 km/h.

c For maximum fuel efficiency, at what speed should the car be driven?

d During the test, the car was driven at 40 km/h for 60 km. How many litres of petrol did it consume?

e The graph is modeled by the formula \( C = \frac{1}{180} s^3 - \frac{2}{3} s + 25 \) for speeds from 20km/h to 80km/h where \( C \) is the petrol consumption in L/100km and \( s \) is the speed in km/h.

i Use this formula to calculate \( C \) at a speed of 90km/h.

ii This formula is a good model for speeds from 20km/h to 80km/h. Why is the formula not useful for \( s = 0 \)?
A company decides to manufacture and sell a new widget. Their marketing research department came up with a cost function and a revenue (income) function.

Cost function \( C = 432 \, 000 - 1800p \)

Revenue function \( R = 6000p - 30p^2 \)

where \( C \) is the cost, \( R \) is the revenue and \( p \) is the selling price of each widget.

a. What type of graph is each function?

b. Copy and complete the table for these two graphs.

<table>
<thead>
<tr>
<th>( p )</th>
<th>0</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( R )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

c. Mark the horizontal 0 to 250 and the vertical axis 0 to 450 000. Draw and label the two functions.

d. i Find the breakeven points (points of intersection).

   ii What do these breakeven points indicate?

e. For which value of \( p \) will a profit be made?

   i \( p = 50 \)  ii \( p = 150 \)  iii \( p = 190 \)

f. What is the loss when \( p = 30? \)

g. At what value of \( p \) does the maximum revenue occur? What is this revenue?

h. At what value of \( p \) does the maximum profit occur? What is this profit?

i. Why are the two answers obtained in parts g and h different?
A cubic equation contains a term in $x^3$. In this course we will only study cubics of the form $y = ax^3$ where $x$ is greater than or equal to 0.

Some examples of simple cubics are $y = x^3$, $y = 5x^3$, $y = \frac{1}{2}x^3$.

Draw the graph of $y = x^3$ for values of $x$ between 0 and 3.

Solution:

First draw up a table of values

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>0.125</td>
<td>1</td>
<td>3.375</td>
<td>8</td>
<td>15.625</td>
<td>27</td>
</tr>
</tbody>
</table>

Now plot these values on a grid.
Exercise 2.2

[There are grids available in the Appendix for you to remove and use.]

1. On the same grid draw and label these curves for $x = 0$ to $x = 3$.
   - $y = x^3$
   - $y = \frac{1}{2} x^3$
   - $y = 1.5 x^3$
   [Use a scale of 0 to 45 on the vertical axis]
   - a How does the coefficient of $x^3$ affect the shape of the curve?
   - b What point do all three curves have in common?

2. The mass, $M$ g, of a cube of cork with side $x$ cm is $M = 0.25 x^3$.
   - a Draw up a table of values for $x$ from 0 to 100 and draw the graph.
   - b Use your graph to find the mass of a cube of cork with side:
     - i 45 cm
     - ii 75 cm
   - c If the mass of a cube of cork is 70 kg, what is its side length?
   - d i Find the side length, in cm, of a cube of cork of volume 1 m$^3$?
     - ii What is the mass, in kg, of this volume of cork?
     - iii Could you lift one cubic metre of cork? Comment.
   - e i What is the mass, in g, of a cube of cork with side 40 cm?
     - ii What is the volume, in cm$^3$, of this amount of cork?
   - f A student compared two cubes of cork, sides 25 cm and 50 cm. He said that the larger cube has a mass twice as big since its side is double that of the smaller cube. Do you agree? Why?

3. The volume of a sphere with radius $r$ is approximately $V = 4.2 r^3$.
   - a Graph this function for $r = 0$ to $r = 15$.
   - What is the volume of a sphere with radius:
     - i 5 cm
     - ii 12.5 cm
   - c If the volume of a sphere is 6000 cm$^3$, what is its radius?
   - d What is the volume, in cm$^3$, of a sphere with radius 10 cm?
   - e What is the volume of a sphere with radius of 9.5 cm?
The equation of a hyperbola is \( y = \frac{a}{x} \) where \( a \) is a constant and \( x \neq 0 \).

Graph the hyperbola \( y = \frac{1}{x} \) for values of \( x \) from 0.2 to 5.

**Solution:**

Draw up a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>2</td>
<td>1</td>
<td>0.7</td>
<td>0.5</td>
<td>0.4</td>
<td>0.3</td>
<td>0.25</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Now plot these values on a grid.

**Note:** As \( x \) increases, the curve gets closer to the \( x \)-axis and as \( y \) increases the curve gets closer to the \( y \)-axis. The graph will never cut the axes since \( y = \frac{1}{x} \) can never equal zero.

The hyperbola actually has two ‘branches’. The graph above is the branch in the first quadrant where \( x \) and \( y \) are positive. The other branch is in the third quadrant where \( x \) and \( y \) are negative.
Exercise 2.3

[There are grids available in the Appendix for you to remove and use.]

1. a  Graph these hyperbolas on the same grid for $x$-values 0.2 to 4.
   
   i  $y = \frac{2}{x}$  
   ii  $y = -\frac{2}{x}$  

   b  How do these two curves differ.

2. To balance a 300kg mass at the end of a plank, a force $F$ kg is applied at a point $dm$ from the pivot point.

   A formula linking these two variables is $F = \frac{600}{d^2}$.

   a  Copy and complete this table.

<table>
<thead>
<tr>
<th>$d$</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>4.0</th>
<th>5.0</th>
<th>6.0</th>
<th>8.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

   b  Plot the points on a grid and draw a smooth curve through them.

   c  Use the graph to find the force, 3.5m from the pivot point, required to balance the mass.

   d  An 85kg man stands on the plank and exactly balances the 300 kg mass. How far is he from the pivot point?

   e  Comment on what happens to the size of the force as the distance from the pivot point decreases.
3 The size of our Sun will differ depending on which planet it is observed from. The table shows the relative distance and width of the Sun as observed from six planets.

<table>
<thead>
<tr>
<th>Planet</th>
<th>Mercury</th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
<th>Jupiter</th>
<th>Saturn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative distance</td>
<td>0·4</td>
<td>0·7</td>
<td>1·0</td>
<td>1·5</td>
<td>5·2</td>
<td>9·6</td>
</tr>
<tr>
<td>Apparent width</td>
<td>2·5</td>
<td>1·4</td>
<td>1·0</td>
<td>0·7</td>
<td>0·2</td>
<td>0·1</td>
</tr>
</tbody>
</table>

a Plot these points on a grid and draw a smooth curve through them. What is the name of the curve?
b Use your graph to determine the apparent width of the Sun from an asteroid at a relative distance of 2.8 units.
c What is the relative distance of a star when the apparent width of the Sun is 0.4?

4 The current, \( I \) amps, through a wire depends on the resistance, \( R \) ohms, across it. A formula linking these two variables is \( I = \frac{20}{R} \).

a Use this formula to complete the table.

<table>
<thead>
<tr>
<th>( R )</th>
<th>1</th>
<th>2</th>
<th>2·5</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b Graph these points on a grid and draw a smooth curve through them.
c Copy and complete the statement:
As the resistance increases the current ______ .
d A piece of resistance wire was connected in the circuit and the current flowing through it was measured at 1·5 amps. What is the resistance of the wire?
A straight road joins two towns 240 km apart. The time it takes to travel between the towns depends on the average speed of travel.

a. Show that the speed, $S$ km/h, and the time taken, $t$ h, are related by the formula $t = \frac{240}{S}$.

b. Draw up a table using $S$ values 20, 30, 40, 60, 80 and 120.

c. Plot these points on a grid and draw a smooth curve through them.

d. What time does it take to cover the distance at 100 km/h?

e. What is the average speed of journey taking $3 \frac{1}{2}$ hours?

f. The legal speed limit is 100km/h but Nigel travels at 120km/h. How much time, in minutes, is saved compared to the time taken when traveling at the maximum legal limit. Do you think that the time saved is worth the risk of Nigel speeding?
2.4 Exponential curves

The last non-linear graphs that we consider are **exponential functions**. The equation of an exponential function is \( y = b(a^x) \) where \( a \) and \( b \) are constants.

Some simple exponential functions are \( y = 2^x \), \( y = 4(3^x) \), \( y = \left(\frac{1}{2}\right)^x \).

Graph the curve \( y = 2^x \) for values of \( x \) between 0 and 4.

**Solution:**

First draw up a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>1</td>
<td>1.4</td>
<td>2</td>
<td>2.8</td>
<td>4</td>
<td>5.7</td>
<td>8</td>
<td>16</td>
</tr>
</tbody>
</table>

Now plot these values on a grid.

**Note:** Use the power key \( x^y \) on your calculator to find the \( y \)-values.
Properties of the exponential curve $y = b(a^x)$

- When $a > 1$, the graph goes up. This models growth, for example, an investment earning compound interest.
- When $0 < a < 1$, the graph goes down. This models decay, for example, a car depreciating in value over time.
- When $b = 1$ the curve will pass through the point $(0, 1)$.
- When $b \neq 1$, the curve will pass through the point $(0, b)$.
- The $x$-axis is a horizontal asymptote. The curve gets closer and closer to the $x$-axis but never quite touches it.

Exponential growth $a > 1$

For example, $y = 2^x$ and $y = 4(3^x)$

Exponential decay $0 < a < 1$

For example, $y = \left(\frac{1}{2}\right)^x$ and $y = 4\left(\frac{1}{3}\right)^x$

Exercise 2.4

[There are grids available in the Appendix for you to remove and use.]

1. By drawing two sketches (not on grid paper) explain why $y = 2^x$ is not the same as $y = x^2$.

2. Draw these curves on the same grid for values of $x$ from 0 to 3.
   a. $y = 2^x$
   b. $y = 3^x$
   a. Comment on the difference between the two curves.
   b. Are these exponential growth or decay curves. Why?
3 Draw these curves on the same grid for values of $x$ from 0 to 4.
   
   i $y = \left(\frac{1}{3}\right)^x$
   ii $y = \left(\frac{1}{2}\right)^x$
   iii $y = \left(\frac{2}{3}\right)^x$

   a Comment on the difference between the three curves.
   b Are these exponential growth or decay curves? Why?

4 Draw these curves on the same grid for values of $x$ from 0 to 3.

   a $y = 2^x$
   b $y = 2^{2x}$
   c $y = 3(2^x)$

   d Comment on the difference between the three curves.

5 Graph the function $y = \frac{1}{2}(4^x)$ for $0 < x < 3$. Use the graph to determine:

   a the value of $y$ when $x = 2.7$
   b the value of $x$ when $y = 10$.

6 The mass of an orang-utan can be expressed using the equation $M = 1.5(1.2)^t$, where $M$ is the mass in kilograms and $t$ is the age in months.

   a Draw a graph for the mass of an orang-utan from birth to 6 months of age.
   b What is the mass of an orang-utan at birth?
   c What is its mass after five and a half months?
   d An orang-utan has a mass of 2 kg. How much time would it take for its mass to double?
   e Is the formula given an exact representation of an orang-utan’s mass or only approximate? Comment.
   f Do you expect the mass of the orang-utan to keep increasing according to the formula given throughout its life? Explain.
2.5 Miscellaneous problems

The following exercise can be done now or used as revision later in the course.

Summary of graph types

- Straight line \( y = mx + b \) or \( ax + by + c = 0 \). [has a term in \( x \)]
- Parabola \( y = ax^2 + bx + c \). [has a term in \( x^2 \)]
- Cubic \( y = ax^3 \). [has a term in \( x^3 \)]
- Hyperbola \( y = \frac{a}{x} \). [\( x \) term in denominator]
- Exponential \( y = b(a^x) \). [\( x \) term as an index]

Exercise 2.5.

1. Use one of the words in the list to describe the shape of each curve.

<table>
<thead>
<tr>
<th></th>
<th>line</th>
<th>parabola</th>
<th>cubic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>exponential</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>( y = 2x - 4 )</td>
<td>( y = 2x^3 )</td>
<td>( y = 3^x )</td>
</tr>
<tr>
<td>b</td>
<td>( y = 2x^3 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>( y = 3^x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>( A = \pi r^2 )</td>
<td>( y = \frac{4}{x} )</td>
<td>( 3x + 2y + 1 = 0 )</td>
</tr>
<tr>
<td>e</td>
<td>( y = \frac{4}{x} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>f</td>
<td>( 3x + 2y + 1 = 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>g</td>
<td>( xy = -2 )</td>
<td>( y = \left( \frac{3}{4} \right)^x )</td>
<td>( y = 2x^3 + 3x - 9 )</td>
</tr>
<tr>
<td>h</td>
<td>( y = \left( \frac{3}{4} \right)^x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>i</td>
<td>( y = 2x^3 + 3x - 9 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>j</td>
<td>( y = 2x^2 + 3x - 9 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2 The formula for the surface area of a cylinder with height 1 unit that is closed at one end is given by the formula \( A = \pi r^2 + 2\pi r \).
Graph this relationship for \( 4 \leq r \leq 6.5 \), taking steps of 0.5, and use the graph to answer the following questions.

a What is the surface area of a cylinder, closed at one end, with radius:

i 4.8cm  
ii 6.3cm 

b What is the radius of the cylinder with surface area:

i 126m²  
ii 156cm² 

3 The number of chirps per minute, \( n \), that a North American cricket makes at \( T^\circ C \) is \( T = \frac{5n}{36} + 4.5 \).

a Copy and complete the following table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b On the grid provided in the Appendix, graph this function for values of \( n \) from 0 to 150.

c Use your graph to answer the following:

i If you hear 72 cricket chirps per minute, what is the approximate temperature?

ii At what temperature does the cricket stop chirping?

iii Is it true to say that as the temperature keeps increasing the number of chirps per minute will also keep increasing? Explain.

4 A stone is thrown upwards from the ground so that after \( t \) seconds its height \( h \) m is given by \( h = 14.7t - 4.9t^2 \).

a Graph this function for \( t \) between 0 and 3 in steps of 0.5.

b From the graph estimate:

i the greatest height reached 

ii the time(s) taken to reach a height of 8m.
5 The wavelength of sound, \( L \) m, is related to its frequency, \( n \) vibrations per second by the formula \( L = \frac{332}{n} \).

a Graph this function for \( 10 \leq n \leq 100 \).

b Use your graph to estimate:
   i the wavelength when \( n = 22 \) vibrations per second
   ii the number of vibrations for a wavelength of 25 m.

6 The number of bacteria, \( N \), in a culture is given by \( N = 100 \left( 3^{0.01t} \right) \) where \( t \) is the time in minutes.

a Draw a graph of this relationship for \( t \) between 0 and 180 minutes in steps of 30 minutes.

b How many bacteria were in the culture initially? [at time \( t = 0 \)]?

c Use your graph to estimate:
   i the number of bacteria after \( 2 \frac{1}{4} \) hours
   ii when the number of bacteria is double the initial population.
Do you understand the following words or expressions?

Look back in your notes, if necessary, and give your explanations here.

parabola___________________________________________________
_________________________________________________________________________________

quadratic function______________________________________________
_________________________________________________________________________________

cubic equation___________________________________________________
_________________________________________________________________________________

hyperbolic function_____________________________________________
_________________________________________________________________________________

e xtrapolate___________________________________________________
_________________________________________________________________________________

interpolate___________________________________________________
_________________________________________________________________________________

exponential curve ___________________________________________
_________________________________________________________________________________

exponential growth __________________________________________
_________________________________________________________________________________

exponential decay ___________________________________________
_________________________________________________________________________________

asymptote___________________________________________________
_________________________________________________________________________________
non-linear relationship
Exercises

[There are grids available in the Appendix for you to remove and use.]

1. On Earth, the formula \( d = 5t^2 \) represents the distance, \( d \) m, that an object falls in terms of the time taken, \( t \) s.
   a. Graph distance against time for the first 6 seconds.
   b. How far does the object fall in:
      i. the first two seconds
      ii. the first four seconds
   c. How far does the object fall in:
      i. the second and third second
      ii. the fifth and sixth second
   d. On the moon, an object falls according to the equation \( d = 0.8t^2 \).
      On the same grid plot this graph using a different colour.
   e. Compared to the moon, how much further can an object fall on Earth in the five seconds?

2. It costs $144 per three-month period to sponsor a child through World Vision. Two people sharing the cost would contribute $72 each and four people would contribute $36 each.
   a. Copy and complete this table where \( n \) is the number of people sharing the cost and \( C \) is the cost per person.

   \[
   \begin{array}{cccccccc}
   n & 1 & 2 & 3 & 4 & 5 & 8 & 10 & 12 & 16 \\
   C & 144 & 72 & 36 & & & & & & \\
   \end{array}
   \]
   b. Plot these points and draw a smooth curve through them.
   c. Why can’t \( n \) have a value of 4.5?
   d. What is the cost per person if 9 people are sharing.
   e. Show that \( C = \frac{144}{n} \) is the equation of your graph.
3 The surface area, $S$, of a sphere can be approximated by the formula $S = 4\pi D^2$ where $D$ cm is its diameter.
   a Graph this function for values of $D$ from 0 to 6.
   b What is the approximate surface area of a sphere with diameter 3.5 cm?
   c What is the diameter of a sphere with surface area $80\text{cm}^2$?

4 The formula $A = P(1 + r)^n$ gives the final amount after investing $P$ for $n$ years at an interest rate $r$, where $r$ is a decimal. When $3000$ is invested at 10% pa ($r = 0.1$) for periods of 0 to 9 years, this formula becomes $A = 3000(1.1)^n$ for $n$ from 0 to 9.
   a Graph this function for $n$ from 0 to 9.
   b Use your graph to estimate the amount accrued after
      i $2\frac{1}{2}$ years
      ii $6\frac{1}{2}$ years.
   c How long does it take for your investment to reach
      i $4500$
      ii $7000$?
   d How long does it take for your initial investment to double?
   e Estimate how long it would take for your initial investment to triple.

5 When a motorist sees danger on the road he must first react to it (reaction distance) and then apply the brakes to stop (braking distance). These two actions constitute stopping distance where:

   Stopping distance = reaction distance + braking distance.

   If reaction distance is $R = 0.7v$ and braking distance is $B = 0.01v^2$:
   a Show that the stopping distance is $S = 0.7v + 0.01v^2$
   b Graph the three equations for $R$, $B$ and $S$, for values of $v$ from 0 km/h to 100 km/h.
   c Comment on the shape of each curve.
d What is the stopping distance at
   i 20 km/h?  
   ii 65 km/h?  
   iii 100 km/h?

e True or false. Doubling the speed:
   i doubles the reaction distance  
   ii doubles the braking distance  
   iii doubles the stopping distance.

f A motorist travelling at 75 km/h sees a small child run out onto the road after a ball. The child is 100m in front of the car. Will the motorist be able to stop in time?

g Explain why many road safety advertisements put emphasis on slowing down.
Student evaluation

When you have finished this unit of work see if you can do these things.

Tick if you can do them with confidence:

- use the quadratic graph to find minimum and maximum values in practical contexts
- generate tables of values to graph cubic, exponential and hyperbolic functions using pen and paper
- recognise that for $a > 1$, $y = b(ax)$ represents exponential growth and for $0 < a < 1$, it represents exponential decay
- use algebraic functions as models of physical phenomena
- recognise the limits of models when interpolating and/or extrapolating

Ask for further help with any you feel unsure about.

Please write your questions and any other comments here and overleaf.
Appendix

Grids for use with Exercise 2.1
Grids for use with Exercise 2.2
Five grids for use with Exercise 2.3
Grids for use with Exercise 2.4
Five grids for use with Exercise 2.5
Five grids for use with end of activity Exercise
Exercise 2.1

1a

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h ) (m)</td>
<td>0</td>
<td>45</td>
<td>80</td>
<td>105</td>
<td>120</td>
<td>125</td>
<td>120</td>
<td>105</td>
<td>80</td>
<td>45</td>
<td>0</td>
</tr>
</tbody>
</table>

c  Height arrow reaches

d  10 seconds
e  125 m
2. a) 

<table>
<thead>
<tr>
<th>d (km)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>h (m)</td>
<td>0</td>
<td>8</td>
<td>32</td>
<td>72</td>
<td>128</td>
<td>200</td>
<td>288</td>
<td>392</td>
<td>512</td>
<td>648</td>
<td>800</td>
</tr>
</tbody>
</table>

b) Distance to horizon

![Graph showing distance to horizon vs height]

c) 42 km  
d) 570 metres

3. a) 6 L/100 km  
b) 8 L/100 km  
c) 60 km/h  
d) 4.2 L  
e) i) 10 L/100km

ii) For \( s = 0 \), the formula gives a consumption of 25 L/100km but if the speed \( s \) is zero, the car is stationary.
4. a) Cost function is a line; revenue function is a parabola.

b) | P  | 0  | 50 | 100 | 150 | 200 |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>432 000</td>
<td>342 000</td>
<td>252 000</td>
<td>162 000</td>
<td>72 000</td>
</tr>
<tr>
<td>R</td>
<td>0</td>
<td>225 000</td>
<td>300 000</td>
<td>225 000</td>
<td>0</td>
</tr>
</tbody>
</table>

c)  


d) i) when p = $80 and $180  
   ii) cost = revenue

e) ii

f) $225 000 (vertical distance between two curves)

g) p = $100; revenue = $300 000

h) p = $130; profit = $75 000

i) The profit is the vertical difference between the revenue function and the cost function. The greatest vertical separation occurs when p = $130. When p = $100, revenue is greater but then costs are greater too.
Exercise 2.2

1

- The steepness of the graph is determined by the coefficient of $x^3$. The greater this coefficient, the faster the curve rises.

- $(0, 0)$
2a

b  i  20 000 g    ii  106 000 g

c  65 cm

d  i  100 cm    ii  250 kg    iii  student response

e  i  16 000 g    ii  64 000 cm$^3$

f  No, doubling each side length increases the mass eight-fold (double $\times$ double $\times$ double). Note the values from the graph.
b  i  525 cm³  ii  8200 cm³  
c  11.3 cm  
d  4200 cm³  e  3600 cm³
Exercise 2.3

1. a

b. They are exactly the same shape but the second curve has negative y-values. The positive sign has the curve above the x-axis and the negative sign puts the reflection of this curve below the x-axis.

2. a

<table>
<thead>
<tr>
<th>d</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>4.0</th>
<th>5.0</th>
<th>6.0</th>
<th>8.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>1200</td>
<td>600</td>
<td>400</td>
<td>300</td>
<td>240</td>
<td>200</td>
<td>150</td>
<td>120</td>
<td>100</td>
<td>75</td>
</tr>
</tbody>
</table>
c \hspace{0.5cm} 170 \text{ kg} \hspace{0.5cm} d \hspace{0.5cm} 7 \text{ m} \hspace{0.5cm} e \hspace{0.5cm} \text{increases}
3 a

Relative distance, $d$

0 1 2 3 4 5 6 7 8 9 10

Apparent width, $w$

0 1 2 3

4 a

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>2.5</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I</td>
<td>20</td>
<td>10</td>
<td>8</td>
<td>5</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0.5</td>
<td>0.4</td>
</tr>
</tbody>
</table>

b 0.4
c 2.8
c current decreases

d 13 ohms

5 a The formula for speed is speed = \( \frac{\text{distance}}{\text{time}} \).

Re-arranging gives time = \( \frac{\text{distance}}{\text{speed}} \). With the distance being 240 km, the formula becomes \( t = \frac{240}{s} \).

<table>
<thead>
<tr>
<th>S</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>t</td>
<td>12</td>
<td>8</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>
Exercise 2.4

1 Student response; but consider \( y = x^2 \) is a parabola while \( y = 2^x \) is an exponential curve and rises more quickly.
a  \( y = 2^x \) rises less steeply than the \( y = 3^x \) curve. The larger the value of \( a \), the faster the curve rises.

b  Exponential growth. Determine not only from their growth (increasing upwards) but also since \( a > 1 \).
a The smaller the value of a, the faster the curve decreases. It begins to flatten out as it approaches the x–axis.

b Exponential decay. Determine not only from their downward trend, but also since $0 < a < 1$ in each case.
All curves have the same value for a. But the value of b is different. As b increases so does the steepness of the curve. When x = 0, the curves cut the y-axis at their respective b values.

\[ y = 2x^2 \]

a \( \approx 2.1 \) or 2.2
b  1·5 kg    c  4·1 kg    d  3·8 months

e  No, only an approximation. The mass of animals don’t correspond to formulas. This formula was probably worked out from examining a large number of cases and arrived at as the equation of the curve of best fit.

f  No. This formula applies to this youthful stage in the growth of orang-utans. A certain point will be reached when the mass of the animal remains fairly well constant.

**Exercise 2.5**

[No graphs are given with these answers. It is assumed that by now you are skilled enough to be able to draw them confidently. Nevertheless you will know your graphs are correct if the answers you obtain from them concur with those given below.]

<table>
<thead>
<tr>
<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
<th>j</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>line</td>
<td>cubic</td>
<td>exponential</td>
<td>parabola</td>
<td>hyperbola</td>
<td>line</td>
<td>hyperbola</td>
<td>exponential</td>
<td>cubic</td>
<td>parabola</td>
</tr>
</tbody>
</table>
Here is a table of values you could use to draw your graph. The area, \( A \), is given correct to the nearest whole number.

<table>
<thead>
<tr>
<th>( r )</th>
<th>4·0</th>
<th>4·5</th>
<th>5·0</th>
<th>5·5</th>
<th>6·0</th>
<th>6·5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>75</td>
<td>92</td>
<td>110</td>
<td>130</td>
<td>151</td>
<td>174</td>
</tr>
</tbody>
</table>

\[ a \quad \text{i} \quad \approx 102 \text{ cm}^2 \quad \text{ii} \quad \approx 165 \text{ m}^2 \]

\[ b \quad \text{i} \quad \approx 5.4 \text{ m} \quad \text{ii} \quad \approx 6.1 \text{ cm} \]

3a

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>4·5</td>
<td>7·3</td>
<td>10</td>
<td>12·8</td>
<td>15·6</td>
<td>18·4</td>
<td>21·2</td>
</tr>
</tbody>
</table>

c

\[ i \quad 14.5°C \quad \text{ii} \quad 4.5°C \]

\[ iii \quad \text{No. The number of chirps will not keep increasing. Common sense tells us that at high enough temperatures the crickets will be killed (cooked crickets). Dead crickets don’t chirp all that well!} \]

4b 11 m; On way up: about 0.7 seconds; on way down: about 2.3 seconds.

5a A suitable table to draw a graph from could be

<table>
<thead>
<tr>
<th>( n )</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>33·2</td>
<td>16·6</td>
<td>11·1</td>
<td>8·3</td>
<td>6·6</td>
<td>5·5</td>
<td>4·7</td>
<td>4·2</td>
<td>3·7</td>
<td>3·3</td>
</tr>
</tbody>
</table>

\[ b \quad \text{i} \quad 15 \text{ m} \quad \text{ii} \quad 13 \text{ vibrations per second} \]

6a A suitable table to draw a graph from could be

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>30</th>
<th>60</th>
<th>90</th>
<th>120</th>
<th>150</th>
<th>180</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>100</td>
<td>139</td>
<td>193</td>
<td>269</td>
<td>374</td>
<td>520</td>
<td>722</td>
</tr>
</tbody>
</table>

b 100

c

\[ i \quad \text{about 440} \quad \left( \text{Note: } 2\frac{1}{4} \text{ hours} = 135 \text{ minutes} \right) \quad \text{ii} \quad \text{about 63 minutes.} \]
AM4 Modelling linear and non-linear relationships

Part 3: Variation
Contents

Introduction........................................................................................................................................3

3.1 Direct variation ..............................................................................................................................4

3.2 More on direct variation ...............................................................................................................7

3.3 Inverse variation ..........................................................................................................................11

Terminology ........................................................................................................................................15

Exercises ............................................................................................................................................17

Student evaluation .............................................................................................................................19

Answers .............................................................................................................................................21
This is the third of three parts covering the syllabus topic AM4 Modelling linear and non-linear relationships, in the Algebraic modelling component of the course.

**Specific content outcomes**

By the end of Part 3, you will have been given opportunities to:

- understand the concept of direct variation
- develop equations such as \( y = ax^2 \) and \( h = at^3 \) from descriptions of situations where one quantity varies directly as the power of another
- develop equations such as \( y = \frac{a}{x} \) from descriptions of situations in which one quantity varies inversely with another
- evaluate the constant of proportionality in the equations given one pair of variables, and using the resulting formula to find other values of the variables

*For students in Distance Education Centres only:*

There is an evaluation page at the back of this part; fill it in when you have finished the work; say how easy/hard/interesting you find this work; ask relevant questions and return your comments to your teacher.
3.1 Direct variation

Consider this table of values. There is a relationship between the two variables $x$ and $y$. In this relationship as $x$ increases so does $y$. In fact, when the $x$ value is doubled, the $y$ value doubles.

We say $y$ varies directly with $x$ or $y$ is proportional to $x$. We write $y \propto x$ or $y = kx$ where $k$ is the constant of proportionality.

In this table, the relationship is $y = 3x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
</tbody>
</table>

When a car travels at a constant speed, the distance travelled, $d$, varies directly with the time taken, $t$. If the car travels 175km in 2.5 hours, find:

1. an equation linking $d$, $t$ and a constant $k$
2. the value of $k$
3. the distance the car travels in 4 hours
4. how long it will take the car to travel 350km.

**Solutions:**

1. $d = kt$

2. Here $d = 175$, $t = 2.5$
   
   $175 = k \times 2.5$
   
   $k = \frac{175}{2.5}$
   
   $k = 70$
3 When $t = 4$, $d = 70 \times 4 = 280\text{km}$

4 When $d = 350$,

$$350 = 70 \times t$$

$$t = \frac{350}{70}$$

$$t = 5\text{h}$$

In an electrical circuit the current, $A$ milliamps, varies directly as the voltage, $V$ volts. When a voltage of 25 volts is applied, the current is 10 milliamps. Find:

1 an equation linking $A$ and $V$.

2 the current through the resistor when the voltage is 30 volts.

3 the voltage when the current through the resistor is 45 milliamps.

Solutions:

1 Here $A = 10$, $V = 25$

$$A = k \times V$$

$$10 = k \times 25$$

$$k = \frac{10}{25} = 0.4$$

So the equation is $A = 0.4V$.

2 When $V = 30$

$$A = 0.4 \times 30 = 12$$

The current through the resistor is 12 milliamps.

3 When $A = 45$

$$45 = 0.4 \times V$$

$$V = \frac{45}{0.4} = 112.5$$

The voltage is 112.5 volts.
Exercise 3.1

1. $P$ varies directly as $Q$.
   a. Write an equation, linking $P$, $Q$ and a constant $k$.
   b. If $Q = 20$ when $P = 15$, find $k$.
   c. Use your equation to find
      i. $P$ when $Q = 48$
      ii. $Q$ when $P = 75$

2. The cost, $C$, of buying petrol varies directly as the quantity, $L$ litres, purchased.
   a. Write an equation linking $C$, $L$ and a constant $k$.
   b. If 25L of petrol cost $23.75, find a value for $k$.
   c. Use your equation to find:
      i. the cost of buying 35L of petrol
      ii. the number of litres of petrol you can buy for $28.50.

3. The number of rotations, $r$, of a flywheel is proportional to the spin time, $t$ minutes.
   a. Write an equation linking $r$, $t$ and a constant $k$.
   b. If the flywheel rotates 1500 times in 2 minutes, find $k$.
   c. Calculate:
      i. the number of rotations the flywheel can make in 20 seconds.
         [Hint: 20 seconds = $\frac{1}{3}$ minute]
      ii. the time it will take the flywheel to make 4125 revolutions.

4. The mass, $m$, of a solid that dissolves in a solution varies directly as the solution’s temperature, $T^\circ C$.
   If 40 grams dissolve at 50$^\circ C$, calculate:
   a. the constant of proportionality $k$
   b. the mass of the solid that will dissolve at 70$^\circ C$
The last section dealt with linear variation. There are other situations where one quantity varies as some power of another quantity.

Consider this table of values.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
</tr>
</tbody>
</table>

There is no simple linear relationship between $x$ and $y$, such as $y = kx$.

Now consider this table of values.

In this table the values of $x$ have been squared and the equation linking $x^2$ and $y$ is $y = 2x^2$.

<table>
<thead>
<tr>
<th>$x^2$</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>18</td>
</tr>
<tr>
<td>16</td>
<td>32</td>
</tr>
<tr>
<td>25</td>
<td>50</td>
</tr>
</tbody>
</table>

We say that $y$ varies directly as the square of $x$ or that $y$ is proportional to the square of $x$. We can write this as $y \propto x^2$.

**Note:** You will generally be told the relationship between two quantities in words and you can then use this to find an equation linking them.
The maximum safe load, $L$, that can be suspended from a steel cable is proportional to the square of the diameter, $d$, of the cable. A cable of diameter 7mm has a maximum safe load of 0.35t.

1. Find an equation linking $L$, $d$ and a constant $k$.

2. What is the maximum safe load for a cable with diameter 10mm? Give answer correct to 2 decimal places.

3. A load of 1.2t is to be lifted by a cable 125mm thick. Can this be done safely with this cable?

**Solutions:**

1. $L$ varies directly as $d^2$. 
   
   $L = kd^2$
   
   $0.35 = k \times 7^2$
   
   $k = \frac{0.35}{7^2}$
   
   $\approx 0.0071428...$

   So $L = 0.007143d^2$

2. When $d = 10$,
   
   $L = 0.007143 \times 10^2$
   
   $= 0.7143$

   So the maximum safe load is 0.71t.

3. When $d = 12.5$cm,
   
   $L = 0.007143 \times 12.5^2$
   
   $= 1.11609375$

   The maximum safe load is 1.1t. Since the load to be lifted is greater than this, it can’t be safely done with this cable.

The time, $T$, it takes for one swing of a simple pendulum (period) varies directly as the square root of its length, $L$. A 2m pendulum takes 2.8 seconds.

1. Find an equation linking $T$, $L$ and a constant $k$.

2. What is the period of a pendulum which is 3m long?

3. If a pendulum has a period of 3.8 seconds, find its length.

**Solutions:**
1

\[ T = k\sqrt{L} \]

\[ 2.8 = k \times \sqrt{2} \]

\[ k = \frac{2.8}{\sqrt{2}} \]

\[ = 1.979898... \]

Equation is \[ T = 1.98\sqrt{L} \]

2

\[ T = 1.98 \times \sqrt{3} \]

\[ \approx 3.43 \text{ s} \]

3

\[ 3.8 = 1.98 \times \sqrt{L} \]

\[ \sqrt{L} = \frac{3.8}{1.98} \]

\[ = 1.91919... \]

\[ L \approx 3.68 \text{ m} \] [square both sides]

**Exercise 3.2**

1. Write an equation for each statement, including a constant \( k \).
   
   a. \( P \) varies directly as the square of \( Q \).
   
   b. \( h \) is proportional to the cube of \( w \).
   
   c. \( m \) varies as the square root of \( n \).

2. If \( x \) varies directly as \( y^2 \) and \( x = 40 \) when \( y = 4 \). Find:
   
   a. the constant of proportionality \( k \)
   
   b. the value of \( x \) when \( y = 5 \)
   
   c. the value of \( y \) when \( x = 30 \)

3. If \( q \) varies directly as the square root of \( r \) and \( q = 10 \) when \( r = 25 \), find:
   
   a. an equation linking these two variables with a constant \( k \).
   
   b. \( q \) when \( r = 36 \)
   
   c. \( r \) when \( q = 12.3 \)
4 The mass of a metallic ball varies as the cube of its diameter. A metallic ball of diameter 5cm has a mass of 350g. Find:

a an equation linking the mass and the diameter of the metallic ball
b the mass of a ball with diameter 6.5cm
c the diameter of a ball with mass 1kg
3.3 Inverse variation

Consider this table of values.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

There is a relationship between the two variables $x$ and $y$. In this relationship as $x$ increases $y$ decreases. In fact, when the $x$ value is doubled, the $y$ value halves.

We say $y$ varies **indirectly** with $x$ or $y$ is **inversely proportional to $x$**.

We write $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$ where $k$ is a constant.

In this table, the relationship is $y = \frac{60}{x}$.

Boyle’s Law states that the volume, $V$, of a gas is inversely proportional to the pressure, $P$, to which the gas is compressed. At a pressure of 100kPa the volume of a gas is 50L. Find:

1. an equation linking $V$ and $P$
2. the volume of gas, to 1 decimal place, at a pressure of 180kPa
3. the pressure, to 1 decimal place, of 60L of the gas

**Solutions:**
1 \[ V = \frac{k}{P} \]
\[ 50 = \frac{k}{100} \]
\[ k = 50 \times 100 = 5000 \]
So equation is \( V = \frac{5000}{P} \).

2 When \( P = 180 \)
\[ V = \frac{5000}{180} = 27.7777... \]
So the volume is 27.8L.

3 When \( V = 60 \)
\[ 60 = \frac{5000}{P} \]
\[ P = \frac{5000}{60} = 83.3333... \]
So the pressure is 83.3kPa.

The force, \( F \), needed to keep an object moving in a circle is inversely proportional to the circle radius, \( r \). For a given object, a force of 20N is needed when the circle radius is 5cm. Find:

1 an equation linking \( F \) and \( r \)
2 the force needed when the radius is increased to 12.5cm
3 the radius if the force is 25N

Solutions:

1 Here \( F = 20 \), \( r = 5 \)
\[ F = \frac{k}{r} \]
\[ 20 = \frac{k}{5} \]
\[ k = 20 \times 5 = 100 \]
So equation is \( F = \frac{100}{r} \).
2 When \( r = 12.5 \)

\[
F = \frac{100}{12.5} = 8
\]

Force required is 8N.

3 When \( F = 25 \)

\[
25 = \frac{100}{r}
\]

\[
r = \frac{100}{25} = 4
\]

Radius of the circle is 4cm.

Exercise 3.3

1 The variables \( x \) and \( y \) vary inversely.

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>8</th>
<th>(iii)</th>
<th>(iv)</th>
<th>32</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td>(i)</td>
<td>20</td>
<td>(ii)</td>
<td></td>
<td>10</td>
<td>8</td>
<td>5</td>
</tr>
</tbody>
</table>

a Find an equation linking the two variables \( x \) and \( y \).

b Use this equation to find the missing values in the table.

2 \( H \) varies indirectly with \( M \). If \( H = 15 \) when \( M = 3 \), find:

a an equation linking the two variables \( H \) and \( M \)

b the value of \( H \) when \( M = 5 \)

c the value of \( M \) when \( H = 10 \)

3 An object of mass \( m \) can be accelerated by applying a force to it. There is an inverse relationship between mass \( m \) and acceleration \( a \). A force applied to a 300kg mass produces an acceleration of 20 m/s\(^2\).

a Find an equation linking mass and acceleration

b What acceleration is generated if the mass is 500kg?

c The acceleration due to gravity on Mars if the period of the pendulum is 3.07s.

4 The period, \( T \) seconds, of a pendulum is inversely proportional to the square root of the acceleration due to gravity, \( g \). On Earth, \( g = 9.8 \) m/s\(^2\). If a pendulum has a period of 1.9s, find:

a an equation linking the two variables, \( T \) and \( g \)

b the period of the pendulum on Jupiter where the acceleration due to gravity is \( g = 26 \) m/s\(^2\).?

c The acceleration due to gravity on Mars if the period of the pendulum is 3.07s.
Do you understand the following words or expressions?

Look back in your notes, if necessary, and give your explanations here.

direct variation______________________________________________

_________________________________________________________

in proportion_______________________________________________

_________________________________________________________

constant of proportionality___________________________________

_________________________________________________________

inverse variation____________________________________________

_________________________________________________________

inversely proportional_______________________________________

_________________________________________________________
1. The speed, \( v \) m/s, of a pulley belt varies directly with the rotational speed, \( n \) revolutions/s, of the pulley wheel. When a wheel is making 15 revolutions per second the belt speed is 16.2 m/s.
   
a. Write an equation linking the variables \( v \) and \( n \).
   
b. Find the belt speed when the wheel is making 12 rev/s.
   
c. What is the maximum rotational speed if the greatest belt speed the system can sustain is 45 m/s.

2. The extension, \( e \), of a cable is proportional to the load, \( L \), suspended from it. If a load of 1.75t on a particular cable produces an extension of 7.35mm, find the extension produced by a load of 2.2t.

3. The distance, \( D \) million light years, of a star galaxy from Earth is directly proportional to its speed, \( S \) in km/s. One star galaxy is 260 million light years away and has a speed of 42 000 km/s.
   
a. Write an equation linking the two quantities \( D \) and \( S \).
   
b. Find the distance of a star galaxy from Earth if it is moving at a speed of 21 600 km/s.

4. The velocity, \( v \), of a body varies as the square root of the applied force, \( F \). A force of 25N produces a velocity of 35 m/s.
   
a. Write an equation linking velocity and force for this body.
   
b. Find the velocity when the applied force is 100N.
   
c. What force is needed to generate a velocity of 45 m/s?

5. The number of eggs used in a cake recipe varies with the square of the diameter of the tin. If two eggs are used in a recipe for a 15cm diameter tin, how many eggs would be used for a 35cm diameter tin?
6. The pressure, $P$, applied to an object is inversely proportional to the contact area, $A$. When the contact area is $5 \text{ cm}^2$, the pressure is $50 \text{ Pa}$.

a. Write an equation linking pressure and area.

b. What is the pressure when the contact area is $40 \text{ cm}^2$?

c. Calculate the contact area for a pressure of $75 \text{ Pa}$.

d. A woman is more likely to leave depressions in linoleum floors when she is wearing high heels than when she is wearing flat-soled shoes. Suggest why.
When you have finished this unit of work see if you can do these things.

Tick if you can do them with confidence:

- understand the concept of direct variation

- develop equations such as $y = ax^2$ and $h = at^3$ from descriptions of situations where one quantity varies directly as the power of another

- develop equations such as $y = \frac{a}{x}$ from descriptions of situations in which one quantity varies inversely with another

- evaluate the constant of proportionality in the equations given one pair of variables, and using the resulting formula to find other values of the variables

Ask for further help with any you feel unsure about.

Please write your questions and any other comments here and overleaf.

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________

________________________________________________________________________
Answers

Exercise 3.1

1 a \( P = kQ \)  
   b \( k = 0.75 \)  
   c i \( P = 36 \)  
   ii \( Q = 100 \)

2 a \( C = kL \)  
   b \( k = 0.95 \)  
   c i \( C = \$33.25 \)  
   ii \( L = 30 \text{ litres} \)

3 a \( r = kt \)  
   b \( k = 750 \)  
   c i \( r = 250 \)  
   ii \( t = 5.5 \text{ minutes} = 5 \text{ min} 30 \text{ sec} \)

4 a \( m = kt \) where \( k = 0.8 \)  
   b \( m = 56 \text{ grams} \)

Exercise 3.2

1 a \( P = kQ^2 \)  
   b \( h = kw^3 \)  
   c \( m = k\sqrt{n} \)

2 a \( k = 2.5 \)  
   b \( x = 62.5 \)  
   c \( y = 3.46 \)

3 a \( q = 2\sqrt{r} \)  
   b \( q = 12 \)  
   c \( r = 37.8225 \)

4 a \( m = 2.8d^3 \)  
   b \( m = 768.95 \text{ g} \)  
   c \( d \approx 7.1 \text{ cm} \)

Exercise 3.3

1 a \( y = \frac{80}{x} \)  
   b (i) 40 (ii) 16 (iii) 10 (iv) 16 (v) 2.5

2 a \( H = \frac{45}{M} \)  
   b \( H = 9 \)  
   c \( M = 4.5 \)

3 a \( a = \frac{6000}{m} \)  
   b \( a = 12 \text{ m/s}^2 \)

4 a \( T = \frac{5.95}{\sqrt{g}} \)  
   b \( T = 1.17 \text{ s} \)  
   c \( g = 3.76 \text{ m/s}^2 \)